SOME BEAUTIFUL FACTS IN MATH

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(1) The little Picard Theorem from complex analysis can be proven using conformal invariance of Brownian Motion: see `https://projecteuclid.org/euclid.aop/1176994888`.

(2) Brownian motion has infinitely many local minima or local maxima. However it has no points which are local maxima on the left and local minima on the right.

(3) If $X+Y$ are independent and sum to a Gaussian, then both are individually Gaussian. The proof is to first observe that X, Y have sub-Gaussian tails and then apply the Hadamard factorization theorem from complex analysis on their Fourier transforms. The same result holds for Poisson variables, up to translation.

(4) Mazur Swindle: if A, B are knots and $A+B$ is the trivial knot, then so are A, B. The reason is that $A = A + (B+A) + (B+A) + \cdots = (A+B) + (A+B) + \cdots = 0$, where 0 is the trivial knot and knot addition is defined by splicing appropriately.

(5) Take a nontrivial knot in $\mathbb{R}^3$. Now go up 1 dimension and form a cone with the original knot as its base. You've constructed a continuous embedding of a 2-d disk into $\mathbb{R}^4$, only non-smooth at the tip of the cone, which is not topologically equivalent to any smooth embedding.

(6) Wigner Semicircle Law: form a random real symmetric matrix with independent, well-behaved entries. Then the eigenvalue distribution will be close to an appropriately rescaled semicircle density.

(7) Dyson Brownian Motion: let a random real symmetric matrix's entries evolve as independent Brownian motions. Then the eigenvalues evolve as repelling Brownian motions.

(8) Borwein integrals: `http://schmid-werren.ch/hanspeter/publications/2014elemath.pdf`

(9) Some combinatorial polytopes are realizable only with irrational coordinates: `https://arxiv.org/pdf/0710.4453.pdf`.

(10) Supermartingale convergence: if you make a sequence of non-positive expected value bets, and your account balance cannot go negative, then your account balance has a long-time limit.

(11) The number of factors of 2 dividing $\sum_{k=1}^{m} \frac{2^k}{k}$ goes to infinity. The reason is that this is the series expansion for the 2-adic logarithm of $1 - 1 = 0$.

(12) Riesz-Thorin interpolation theorem: if a linear operator is bounded from $L^p \to L^p$ and $L^q \to L^q$ then it is also bounded from $L^r \to L^r$ for r in between p and q. One beautiful proof uses the Hadamard three-lines theorem from complex analysis.

(13) FKG inequality: any two increasing functions on a hypercube are positively correlated. For example, if I tell you that a $G(n, 1/2)$ graph has a hamiltonian cycle, it is now more likely to have triangles.

(14) Happy-ending problem: for any n, given enough points in the plane, you must be able to form a convex n-gon.

(15) Muntz-Satz theorem: an increasing sequence $0 = \lambda_1 < \lambda_2 < \dots$ of real numbers yields a set $\{x^{\lambda_i}\}$ with dense span in $C([0,1])$ iff $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty$.

(16) The Hamburger moment theorem says that a sequence $(a_k)_{k\geq 0} \in \mathbb{R}^{\mathbb{N}}$ of hypothetic moments for a real random variable $a_k = \mathbb{E}[X^k]$ is realizable iff it assigns positive expectation to every polynomial which is always positive. A corollary is that if (a_k) is achivable, then so is (a_k^2). Can you show this explicitly?

(17) Bernstein's theorem states that a *completely monotone* function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(-1)^k f^{(k)}(t) \geq 0$ for all $t \geq 0$ and $k \in \mathbb{Z}^+$ is a positive combination of decaying exponential functions, i.e.

$$f(t) = \int_0^\infty e^{-tx} d\mu(t)$$

for some positive measure μ. The proof goes by applying the infinite dimensional Krein-Milman theorem, which states that a compact, convex set in any (Hausdorff locally convex) topological vector space equals the closed convex hull of its extreme points. Indeed the set of completely monotone functions is a convex cone, and the relation

$$f(t) = \Big[f(t) - f(t+\varepsilon)\Big] + f(t+\varepsilon)$$

lets us write any completely monotone f as a positive combination of two other completely monotone functions. (It's not hard to check that both terms are completely monotone.) For f to be an extreme point these two functions must be proportional, which eventually shows that all extreme points are of the form $f(t) = e^{-at}$. In fact this proof does not even require any smoothness of f; the derivative conditions can be replaced with finite differences.

(18) Hurwitz automorphism theorem: a compact Riemann surface of genus g has at most $84(g-1)$ automorphisms.

(19) Need to take a translation-invariant average over $\mathbb{R}^n$? If you're willing to use the Axiom of Choice and drop countable additivity, you can use an invariant mean. This works on any (any!) abelian group.

(20) There exist measure zero sets in the plane containing a unit line segment in every direction. These are called Besicovitch sets.

(21) Fourier multiplication by the indicator of a polyhedron is bounded on all L^p spaces with $1 < p < \infty$. But for a ball, it is unbounded on L^p for all $p \neq 2$. The proof due to Fefferman uses Besicovitch sets and can be found in section 9.3 here: `https://www.math.stonybrook.edu/~bishop/classes/math324.F15/book1Dec15.pdf`.

(22) Let (X, μ_X) be a complete separable metric (Polish) space equipped with a measure with no atom. As a topological space with a Borel measure, (X, μ_X) is isomorphic to Lebesgue measure on the unit interval.

(23) Let A, B be sets in $\mathbb{R}^d$ with given volumes. Then the volume of $A+B$ is minimized when A, B are dilations of the same convex set.

(24) Let $K_1, \ldots, K_n$ be convex sets in $\mathbb{R}^d$. For non-negative t_i, let $f(t_1, \ldots, t_n)$ be the volume of the Minkowski sum $\sum_i t_i K_i$. Then f is a polynomial.

(25) If f, g are rational functions such that $f(x)e^{g(x)}$ has an elementary integral, then it must be $r(x)e^{g(x)}$ for some rational function r. This lets you show that $\int e^{x^2} dx$ is not elementary and make up your own examples. Check out `http://www.cs.ru.nl/~freek/courses/mfocs-2012/risch/Integration%20in%20Finite%20Terms--Maxwell%20Rosenlicht.pdf` and `http://math.stanford.edu/~conrad/papers/elemint.pdf` for some more details.

(26) Apery proved that $\zeta(3)$ is irrational. The proof is completely absurd: `http://pracownicy.uksw.edu.pl/mwolf/Poorten_MI_195_0.pdf`.

(27) Try to make a random walk so that the probability mass function of your position k steps in the future always has values bounded by $f(k)$. If $\sum_j \frac{f(j)}{j} < \infty$ then this is possible, otherwise it is not. These are called unpredictable paths: https://www.math.upenn.edu/~pemantle/papers/EIT.pdf.

(28) Power of two choices: if you drop many balls sequentially into n bins uniformly at random, the fluctuations in the bin sizes will go to infinitely. However if each balls looks at two random bins and uses the less occupied one, the fluctations stay stochastically bounded by $O(\log\log n)$.

(29) Coupling from the past lets you sample *exactly* from many distributions such as Ising models in finite time using a Markov chain.

(30) Arctic circle theorem: a uniformly random domino tiling of a diamond looks like https://cp4space.files.wordpress.com/2014/10/arctic-circle.png. Also the number of such tilings is exactly $2^{\binom{n}{2}}$ where n is the side length of the diamond.

(31) The hitting time of a Borel set by a Brownian motion is measurable.

(32) The mean, median, and mode of any binomial random variable differ by at most 1. If the mean is an integer, all 3 coincide.

(33) Suppose that $A(x) = \sum_{i\leq n} a_i x^i$ has all negative real roots, and the same for $B(x) = \sum_{i\leq n} b_i x^i$. Then $C(x) = \sum_{i\leq n} a_i b_i x^i$ also has all negative real roots.

(34) Any Borel measure on $\mathbb{R}^d$ is regular, meaning that any Borel set can be approximated from above by open sets, and from below by compact sets. A proof is to simply say that the class of such approximable sets always contains the open sets, and is always a σ-algebra.

(35) The (first) Ray-Knight theorem: run a Brownian motion $B(t)$ from $B(0) = 1$ until it hits 0 at time τ. Then consider the pushforward of the uniform measure on $[0,\tau]$ by $B(\cdot)$. This measure has a continuous density, and when restricted to $[0,1]$ this density is distributed as the square of the magnitude of a two-dimensional Brownian motion started at the origin.

(36) Consider a branching Brownian motion on $\mathbb{R}$, meaning that every particle splits into two particles after Poisson amounts of time. Then every particle eventually has a descendant in the lead.

(37) The cover time of a graph (the amount of time needed for a random walk to hit every vertex) is within a constant factor of the expected maximum of a Gaussian free field on the graph. If all hitting times are small compared to the cover time,

then the proportionality constant is 1.

(38) Steiner and Poncelet porism: `https://en.wikipedia.org/wiki/Steiner_chain` and `https://en.wikipedia.org/wiki/Poncelet%27s_closure_theorem`.

(39) It takes $\frac{3}{2}\log_2(n)+O(1)$ shuffles to randomize a deck of n cards: `https://statweb.stanford.edu/~cgates/PERSI/papers/bayer92.pdf`.

(40) A cool technique for proving geometric inequalities is Steiner symmetrization. This takes a general shape and symmetrizes it to a rounder shape with the same volume. For instance you can prove the isoperimetric inequality by symmetrizing until you get a sphere. See `http://www.math.utah.edu/~treiberg/Steiner/SteinerSlides.pdf` for a nice introduction.

(41) Did you ever notice that planetary orbits come back to the same spot every revolution? Apriori, there is no reason this should be true for a radially symmetric potential. The inverse square law is special and has an extra conserved quantity in addition to energy which makes this happen.

(42) Tennis racket theorem: an object in $\mathbb{R}^3$ has three orthogonal principal axes. Rotation around the long and short axes are stable states, but the not for the middle one. This video shows what happens: `https://www.youtube.com/watch?v=r-TnCMZF3fA`.

(43) There is a (correct) Quantum Zeno paradox where constantly measuring something freezes the time evolution.

(44) The wave equation has different behavior in even/odd dimensions. This is why in our 3d world you hear a sound wave for just 1 instant. By contrast, pond ripples don't act this way.

(45) You cannot hear the shape of a drum, but you can hear its dimension, volume, and surface area via Weyl's law.

(46) A spherical triangle has area $A + B + C - \pi$. Also, the law of sines holds if you replace the side lengths with the sines of their subtended spherical arcs.

(47) The reciprocal of a never-zero function on the circle with ℓ^1 Fourier transform also has ℓ^1 Fourier transform. The proof goes by showing that in the Banach algebra of ℓ^1-Fourier functions (under pointwise multiplication), the only characters (non-trivial homomorphisms to $\mathbb{C}$) are the evaluation-at-some-point functionals, and applying a general theorem saying that if all your character values are non-zero

then you are invertible.

(48) Monsky's Theorem says it is impossible to cut a square into an odd number of triangles of equal area. The proof uses an extension of the 2-adic valuation to the real numbers.

(49) Luroth's theorem says that every field between $\mathbb{C}$ and $\mathbb{C}(x)$ is $\mathbb{C}(f(x))$ for some rational function f.

(50) If a monic polynomial of degree n takes values $e^{o(n)}$ on the whole unit circle, then almost all roots are near the unit circle, and their angles are approximately equidistributed.

(51) John's theorem: all norms on $\mathbb{R}^d$ are equivalent to some Euclidean norm up to a $\sqrt{d}$ factor. Both ℓ^1 and ℓ^∞ achieve this bound.

(52) Wilson's oddness theorem: a game with generic payoffs has an odd number of Nash equilibria with probability 1. To construct a two-player game with n equilibria for any n, simply have each player pick a number in $[n]$ where the payoff for both is the smaller number. Then (k,k) for $k \leq [n]$ are exactly the Nash equilibria.

(53) An arbitrary union of closed balls of positive radius is Lebesgue measurable.

(54) Delaunay triangulations satisfy a bunch of nice properties and also look pretty: `https://en.wikipedia.org/wiki/Delaunay_triangulation`.

(55) Effective Aumann Agreement: if two people have common knowledge of their rationality and a common prior, simply taking turns saying their new value for $\mathbb{P}[A]$ for a fixed event A will lead to convergence. Indeed, from the point of view of an outside observer, the sequence of numbers uttered is a martingale bounded in $[0,1]$.

(56) Anyone would guess that nth roots of positive integers are linearly independent over $\mathbb{Q}$ except for the obvious redundancies. I used to think this was a hard theorem because I had never seen a proof. Here is an extraordinarily simple proof. Note that any hypothetical linear relation can be multiplied to have the term 1 appear in reduced form. For example, we can change $\sqrt{2}+\sqrt{10}-\sqrt{27}=0$ into $1+\sqrt{5}-\sqrt{27/2}=0$. But now when we take the trace over $\mathbb{Q}$, 1 is the only term giving a non-zero contribution, which is impossible!