Convex Geometry

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1 Definitions

Definition 1. A set $S \subseteq \mathbb{R}^n$ is *convex* if $a, b \in S$ implies $\lambda a + (1-\lambda)b \in S$ for all $\lambda \in [0,1]$.

Definition 2. A set $K \subseteq \mathbb{R}^n$ is *compact* if it is closed and bounded.

Definition 3. A *convex combination* of points (x_i) is a linear combination $\sum_{i=1}^{N} \alpha_i x_i$ where the α_i are non-negative and add to 1.

Definition 4. The *convex hull* of a set S is the smallest convex set containing S, or equivalently the set of convex combinations of points in S.

Definition 5. A *convex body* in $\mathbb{R}^n$ is a compact, convex set which is not contained in a lower-dimensional affine subspace.

Definition 6. Given a convex body $K \subseteq \mathbb{R}^n$, a point $k \in K$ is an *extreme point* of K if there is no line segment contained in K which contains p on the inside.

Definition 7. A *polytope* is a finite intersection of closed half-spaces; these are common examples of convex bodies.

2 Fundamentals of Convex Sets

1. (Radon's Theorem) Let A be a set of at least $n+2$ points in $\mathbb{R}^n$. Show that A can be partitioned into sets $A = A_1 \cup A_2$ with intersecting convex hulls.

2. (Caratheodory's Theorem) If $X \subseteq \mathbb{R}^n$ and y is in the convex hull of X, show that y is a convex combination of some $n+1$ points in X.

3. (Helly's Theorem) Let $C_1, ..., C_m$ be convex sets in $\mathbb{R}^n$ with $m \geq n+1$. Suppose that every $n+1$ sets have non-empty intersection. Show that all m sets have non-empty intersection. Show that this holds for an infinite family $C_1, ...$ if the C_i are assumed compact, but not otherwise.

4. (Hahn-Banach Theorem) Given two non-intersecting convex bodies $A, B \subseteq \mathbb{R}^d$, show that there is a hyperplane H which separates them.

5. (Krein-Milman Theorem) Let P be a convex body. Show that P is the convex hull of its set of extreme points.

3 Linear Duality, based on Terry Tao's blog

The Farkas lemma is tricky but pretty fundamental. We should probably talk about it.

1. (Farkas Lemma) Let $P_1, ..., P_k : \mathbb{R}^n \to \mathbb{R}$ be affine functions

$$P_i(x_1, ..., x_n) = c_i + \sum_{j=1}^{n} a_{i,j} x_j.$$

   Show that exactly one of the following holds:

   (a) There exists $x \in \mathbb{R}^n$ with $P_i(x) \geq 0$ for all i.

   (b) There exist non-negative reals $q_1, ..., q_n$ such that for all x we have

$$\sum_{i=1}^{n} q_i P_i(x) = -1.$$

   This means that given some linear system of inequality constraints $P_i(x) \geq 0$, either they are solvable, or there is some linear obstruction to solvability.

2. (Optional) If you've seen them, formulate strong duality and max-flow/min-cut in terms of the Farkas lemma.

3. (Hahn-Banach Theorem again; use the Farkas lemma this time!) Given two non-intersecting convex bodies $A, B \subseteq \mathbb{R}^d$, show that there is a hyperplane H which separates them.

4 Problems

1. Show that a compact set K in $\mathbb{R}^d$ has a unique smallest closed ball B containing it. Show that B is minimal iff the convex hull of $K \cap \partial B$ contains the center of B.

2. Let $K \subseteq \mathbb{R}^n$ be compact. Show that the convex hull of K is also compact.

3. Consider the convex set of $n \times n$ *doubly stochastic matrices*, in which all entries are in $[0, 1]$ and every row/column sums to exactly 1. Show that this set is convex and find the set of extreme points.

4. (Famous, heard from Max Schindler) Bob places m points in $\mathbb{R}^n$. Alice picks another point in $\mathbb{R}^n$ and Bob will pick a half-space containing that point. Alice wants to maximize the number of points in that half-space while Bob wants to minimize it. How many points can Alice guarantee?

5. (Based on confusion from the first time I taught this) Suppose that in the definition of the convex hull for a bounded set S, we allow infinite convex combinations $\sum_{i=1}^{\infty} \alpha_i x_i$ as long as we still have $\alpha_i \geq 0$ and $\sum_i \alpha_i = 1$. Show that this is equivalent to the usual definition which only allows finite sums.

6. (John's Theorem) Show that if $K \subseteq \mathbb{R}^n$ is a convex body, then for some affine map T we have $B_1(0) \subseteq T(K) \subseteq B_n(0)$, where $B_r(0)$ is the ball of radius r around 0. Also show that if K is symmetric then we can replace B_n by $B_{\sqrt{n}}$. Show this is tight.

7. (`https://math.stackexchange.com/q/52833`) Let $K \subseteq \mathbb{R}^n$ be a convex body. Show that there is an "approximate center" point $k \subseteq K$ such that for any line segment $[u, v] \subseteq K$ connecting boundary points u, v and containing k, we have $\frac{|u-k|}{|u-v|} \leq \frac{n}{n+1}$. Show this is tight.

8. (Jung's Theorem) Show that a set $S \subseteq \mathbb{R}^d$ of diameter 1 is contained in a ball of radius $\sqrt{\frac{d}{2d+2}}$ and show this is tight.

9. (Happy Ending Problem; different flavor from the rest if you want to do normal combinatorics) Show that for every n there exists N such that for any N distinct points in the plane with no 3 collinear, some n are in convex position.

10. (Kirszbraun Extension Theorem) Suppose $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ satisfies

$$|f(x) - f(y)| \leq |x - y|$$

for all $x, y \in A$. Show that f extends to $F : \mathbb{R}^n \to \mathbb{R}^m$ with the same condition:

$$|F(x) - F(y)| \leq |x - y|$$

but now for all $x, y \in \mathbb{R}^n$.

11. (Matching Polytope; learned from Jenny Iglesias) The motivation for this problem is to define fractional matchings. Consider a finite graph G with an even number of vertices and consider weight functions $f : E(G) \to [0, 1]$ on the edge set such that the total weight of any vertex's neighbors is 1. Suppose additionally that for any odd-size set A of vertices, the edges between A and A^c have total weight at least 1. Show that the space of such functions f is a convex set, and is in fact the convex hull of the set of matchings of G. Also, find an example showing that this is false without the final condition on the weight of edges between A, A^c.

12. (Analysis Problem, apparently from Math 55) Let (x_i) be a sequence of vectors in $\mathbb{R}^n$. Suppose the series $\sum_{i=1}^{\infty} x_i$ is *conditionally convergent*, meaning that the limit of the partial sums exists in $\mathbb{R}^n$ but

$$\sum_{i=1}^{\infty} |x_i| = \infty.$$

Assume further that for any orthogonal projection map P onto a line ℓ through the origin we have

$$\sum_i |P(x_i)| = \infty.$$

Then show that for any $y \in \mathbb{R}^n$ there is a rearrangement of the series $\sum_{i=1}^{\infty} x_i$ which converges to y. Show that the last assumption about projections is necessary. (The case $n = 1$ is called the Riemann Rearrangement Theorem and is pretty famous. It's also easier, so probably do it before doing higher n if you haven't seen it before.)

13. (Alternate Solution to December TST 2017) The Krein-Milman theorem is true in infinite dimensions with "convex body" meaning "compact, convex set"[1]. Consider a function

$$f : \mathbb{R}^2 \to [0, 1]$$

satisfying

$$f(x, y) = \frac{f(x-1, y) + f(x, y-1)}{2}.$$

Assuming that the word "compact" won't cause problems (which it won't), use the Krein-Milman theorem to show that f must be constant.

[1] However the definition of compact not quite "closed and bounded" in infinite dimensions.