Farey Sequences

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In this handout, we'll explore the Farey sequences; the nth Farey sequence F_n results from putting reduced fractions in the unit interval with denominator at most n in order. Surprisingly, lots of cool things happen! Let's look at examples of the first few.

1. Spend a few minutes looking at the fractions arranged below, finding as many patterns as you can. Try proving them.

$\frac{0}{1}$	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	$\frac{1}{1}$
$\frac{0}{1}$	–	–	–	–	–	–	–	–	–	–	$\frac{1}{2}$	–	–	–	–	–	–	–	–	–	–	$\frac{1}{1}$
$\frac{0}{1}$	–	–	–	–	–	–	$\frac{1}{3}$	–	–	–	$\frac{1}{2}$	–	–	–	$\frac{2}{3}$	–	–	–	–	–	–	$\frac{1}{1}$
$\frac{0}{1}$	–	–	–	–	$\frac{1}{4}$	–	$\frac{1}{3}$	–	–	–	$\frac{1}{2}$	–	–	–	$\frac{2}{3}$	–	$\frac{3}{4}$	–	–	–	–	$\frac{1}{1}$
$\frac{0}{1}$	–	–	–	$\frac{1}{5}$	$\frac{1}{4}$	–	$\frac{1}{3}$	–	$\frac{2}{5}$	–	$\frac{1}{2}$	–	$\frac{3}{5}$	–	$\frac{2}{3}$	–	$\frac{3}{4}$	$\frac{4}{5}$	–	–	–	$\frac{1}{1}$
$\frac{0}{1}$	–	–	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	–	$\frac{1}{3}$	–	$\frac{2}{5}$	–	$\frac{1}{2}$	–	$\frac{3}{5}$	–	$\frac{2}{3}$	–	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	–	–	$\frac{1}{1}$
$\frac{0}{1}$	–	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	–	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	–	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	–	$\frac{1}{1}$
$\frac{0}{1}$	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{1}{1}$

Once you've found some patterns and spent a couple minutes trying to prove them, or just given up, turn the page.

Here are two patterns you might have found:

Fact 1: Fix (a, b, c, d). There exists n such that the fractions $\frac{a}{b} < \frac{c}{d}$ are adjacent in F_n if and only if $bc - ad = 1$.

Fact 2: If $\frac{a}{b} < \frac{c}{d}$ are adjacent in F_n, then the first fraction to be inserted between them is $\frac{a+c}{b+d}$, which is already in lowest terms. (Given fractions $\frac{a}{b}, \frac{c}{d}$, the fraction $\frac{a+c}{b+d}$ is called the *mediant*.)

2. Prove the above facts.

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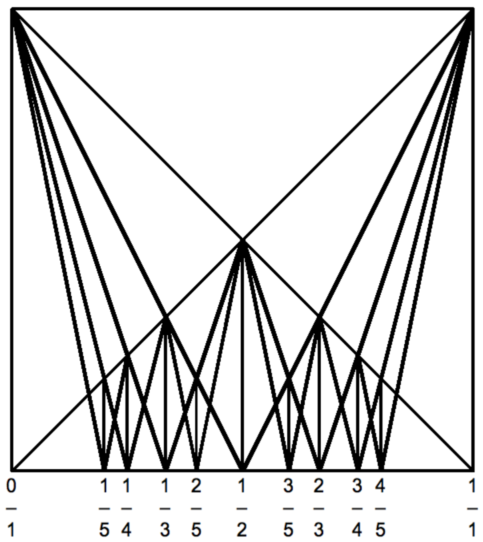
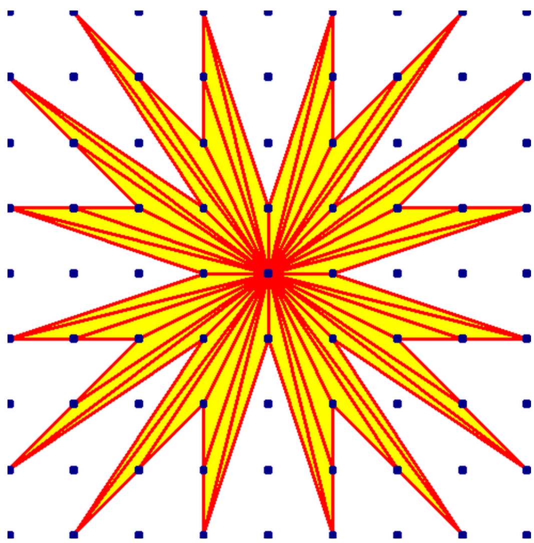
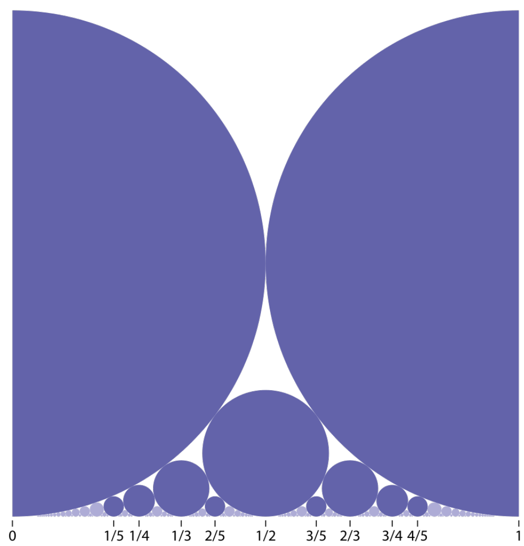
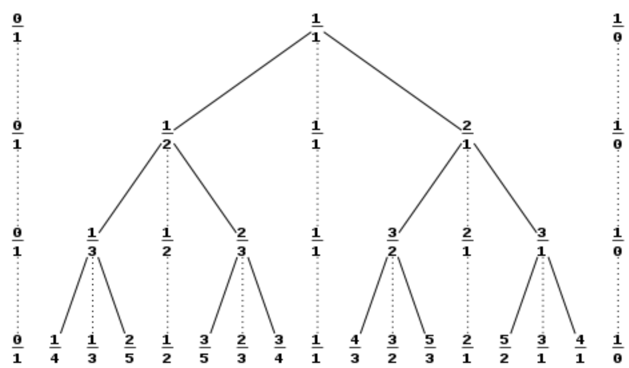
3. Think about these cool pictures. Try to figure out what's happening in each one.









Problems

4. Show that if $\frac{a}{b}, \frac{e}{f}, \frac{c}{d}$ are reduced, consecutive terms of F_n, then $\frac{e}{f} = \frac{a+c}{b+d}$. (Be careful, this is not equivalent to fact 2!)

5. Write F_n in increasing order as $(\frac{a_0}{b_0}, \frac{a_1}{b_1}, \dots \frac{a_m}{b_m})$. Find $\sum_{i=0}^{m-1} \frac{1}{b_i b_{i+1}}$.

6. (Yields a MOP 2014 problem) Find an equivalent formulation of problem 5 that makes no reference to Farey sequences at all, by characterizing what pairs of numbers appear together as adjacent denominators.

7. (Google Code Jam World Finals 2016) For an irrational $\alpha \in (0, 1)$, put a circle of radius α around each non-zero lattice point $(x, y) \neq (0, 0)$ in the plane. In terms of α, which circles are visible for an observer at $(0, 0)$? (A circle is visible if there exists a point p on it that is visible, meaning the line segment from $(0, 0)$ to p does not intersect any other circle.)

8. (Niven and Zuckerman[1]) Show that for any irrational number α, there are infinitely many rationals $\frac{a}{b}$ such that $|\alpha - \frac{a}{b}| \leq \frac{1}{b^2}$. Show that we can in fact replace $\frac{1}{b^2}$ with $\frac{1}{b^2\sqrt{5}}$, and that $\frac{1}{\sqrt{5}}$ is the best possible constant.

9. Letting $|F_n|$ be the number of terms in F_n, show that

$$\lim_{n\to\infty} \frac{|F_n|}{n^2} = \frac{3}{\pi^2}.$$

It would be nice to be completely rigorous and not just write down some infinitary expression without full justification, but this isn't really emphasized in high school so don't worry about it unless you want to.

But in case you want to try, you can show that

$$\frac{|F_n|}{n^2} = \frac{3}{\pi^2} + O\left(\frac{\log n}{n}\right).$$

10. (USA TSTST 2013) A finite sequence of integers $a_1, \dots a_n$ is called *regular* if there exists a real number x such that for $1 \leq k \leq n$,

$$\lfloor kx \rfloor = a_k$$

Given a regular sequence of integers $a_1, a_2, \dots a_n$, for $1 \leq k \leq n$ we say that the term a_k is *forced* if the following condition holds: the sequence

$$a_1, a_2, \dots a_{k-1}, b$$

is regular if and only if $b = a_k$. Find the maximum possible number of forced terms in a regular sequence with 1000 terms.

11. (HMIC 2015) Let $m \geq n$ be positive integers. Let S be the set of pairs (a, b) of relatively prime positive integers such that $a, b \leq m$ and $a + b > m$.

For each pair $(a, b) \in S$, consider the nonnegative integer solution (u, v) to the equation $au - bv = n$ with $v \geq 0$ minimal, and let $I(a, b)$ denote the (open) interval $(\frac{v}{a}, \frac{u}{b})$.

Prove that $I(a, b) \subseteq (0, 1)$ for every $(a, b) \in S$, and that any fixed irrational number $\alpha \in (0, 1)$ lies in $I(a, b)$ for exactly n distinct pairs $(a, b) \in S$.

[1] This is a pretty nice introductory number theory book.

12. (Putnam 2014) Let $f\colon [0,1] \to \mathbb{R}$ be a function, and suppose that there exists $K > 0$ with $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in [0,1]$ (this property means that f is *Lipchitz*). Suppose also that for each rational $r \in [0,1]$, there exist integers a, b such that $f(r) = a + br$. Prove that there exist finitely many intervals $I_1, \dots I_n$ such that f is linear on each I_i and
$$[0,1] = \bigcup_{i=1}^{n} I_n$$

13. (Adaptation of USA TST 2004; also a 2015 RSI project) Let L_n be the square set of lattice points (a,b) with $0 \le a, b \le n$. Let a_n be the maximum possible number of points in L_n that the graph of a strictly convex function [2] $f\colon \mathbb{R} \to \mathbb{R}$ could pass through, subject to the conditions that $f(0) = 0, f(n) = n$. Show that
$$\lim_{n\to\infty} \frac{(a_n)^3}{n^2} = \frac{27}{\pi^2}$$

14. (Related to ISL 2017 C8) In the previous problem, find the limiting shape of the optimal curve, if you zoom out so that it's a function $f : [0,1] \to [0,1]$.

15. (*Topology of Numbers*[3]) The bottom-right picture on the second page of this handout is called the *Stern-Brocot tree.* If you didn't figure out how it works in problem 3, do so now. Then, find a relationship between the position of a rational number in the Stern-Brocot tree and the number's continued fraction representation.

16. (Dhroova Aiylam) Suppose we generate modified Farey sequences starting with positive fractions $\frac{a}{b} < \frac{c}{d}$ by repeatedly taking mediants, and now reducing to lowest terms when possible. (Hence the mediant of $\frac{1}{3}, \frac{1}{7}$ becomes $\frac{1}{5}$.) Show that all reduced fractions $\frac{e}{f} \in [\frac{a}{b}, \frac{c}{d}]$ appear in this modified Farey sequence at some level.

17. (IMO 2013) Let $n \ge 2$ be an integer. Consider all circular arrangements of the numbers $0, 1, \dots n$; the $n+1$ rotations of an arrangement are considered to be equal. A circular arrangement is called *beautiful* if, for any four distinct numbers $0 \le a, b, c, d \le n$ with $a + c = b + d$, the chord joining numbers a, c does not intersect the chord joining b, d. Let M be the number of beautiful arrangements of $0, 1, \dots n$. Let N be the number of pairs (x, y) of positive integers such that $x + y \le n$ and $\gcd(x,y) = 1$. Prove that $M = N + 1$.

18. (ISL 2013) Let ν be an irrational positive number, and let m be a positive integer. A pair of (a,b) of positive integers is called *good* if
$$a \lceil b\nu \rceil - b \lfloor a\nu \rfloor = m.$$
A good pair (a,b) is called *excellent* if neither of the pair $(a-b, b)$ and $(a, b-a)$ is good.

Prove that the number of excellent pairs is equal to the sum of the positive divisors of m.

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[2] Meaning $\frac{f(x)+f(y)}{2} > f(\frac{x+y}{2})$ for all $x > y$

[3] This is an unpublished online introductory number theory book by Allen Hatcher, who is better known for his algebraic topology textbooks. It's short and easy to read, and there's lots more cool stuff relating Farey sequences to continued fractions. I'd definitely recommend taking a look at `http://www.math.cornell.edu/~hatcher/TN/TNbook.pdf` if you liked this handout.