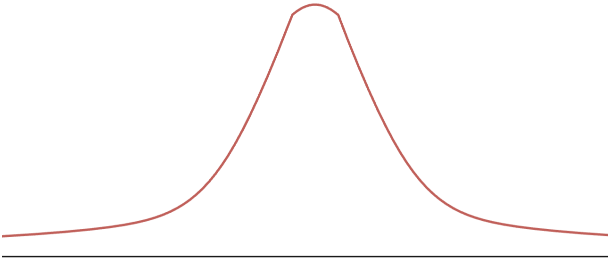
Confinement of Unimodal Distributions in High Dimension and an FKG-Gaussian Correlation Inequality

Mark Sellke (Harvard)

Stanford Probability Seminar
May 20, 2024

Probability distributions take many forms. Which are simplest?

In 1-dimension, the **unimodal** distributions form a very nice class.



Reminders on Log-Concavity

A strict notion of unimodality is **log-concavity**. $d\mu(x) = \rho(x)dx$ is:

- **Log-concave** if $\log \rho(x)$ is concave.
- M-**log-concave** if for positive-definite M and all $x \in \mathbb{R}^N$:

$$\nabla^2 \log \rho(x) \preceq -M \prec 0.$$

Reminders on Log-Concavity

A strict notion of unimodality is **log-concavity**. $\mathrm{d}\mu(x) = \rho(x)\mathrm{d}x$ is:

- **Log-concave** if $\log \rho(x)$ is concave.
- M-**log-concave** if for positive-definite M and all $x \in \mathbb{R}^N$:

$$\nabla^2 \log \rho(x) \preceq -M \prec 0.$$

M-log-concave distributions are **dominated** by the corresponding Gaussian $\gamma_M = \mathcal{N}(0, M^{-1})$.

Reminders on Log-Concavity

A strict notion of unimodality is **log-concavity**. $d\mu(x) = \rho(x)dx$ is:

- **Log-concave** if $\log \rho(x)$ is concave.
- M-**log-concave** if for positive-definite M and all $x \in \mathbb{R}^N$:

$$\nabla^2 \log \rho(x) \preceq -M \prec 0.$$

M-log-concave distributions are **dominated** by the corresponding Gaussian $\gamma_M = \mathcal{N}(0, M^{-1})$.

- Covariance bound [Brascamp-Lieb]:

$$\mathbb{E}^{x\sim\mu}[xx^\top] \preceq \mathbb{E}^{x\sim\gamma_M}[xx^\top] = M^{-1}$$

Reminders on Log-Concavity

A strict notion of unimodality is **log-concavity**. $d\mu(x) = \rho(x)dx$ is:

- **Log-concave** if $\log \rho(x)$ is concave.
- M-**log-concave** if for positive-definite M and all $x \in \mathbb{R}^N$:

$$\nabla^2 \log \rho(x) \preceq -M \prec 0.$$

M-log-concave distributions are **dominated** by the corresponding Gaussian $\gamma_M = \mathcal{N}(0, M^{-1})$.

- Covariance bound [Brascamp-Lieb]:

$$\mathbb{E}^{x \sim \mu}[xx^\top] \preceq \mathbb{E}^{x \sim \gamma_M}[xx^\top] = M^{-1}$$

- μ inherits functional inequalities from γ_M [Bakry-Emery].
  - Spectral gap, isoperimetry, concentration.

Unimodality without Concentration

Unfortunately, some unimodal distributions do not behave so nicely.
Consider the two-component Gaussian mixture

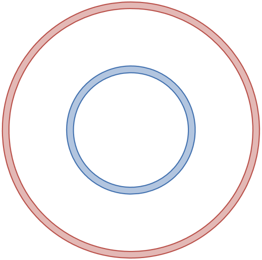
$$\frac{1}{2}\mathcal{N}(0, I_N) + \frac{1}{2}\mathcal{N}(0, 4\, I_N).$$

Unimodality without Concentration

Unfortunately, some unimodal distributions do not behave so nicely. Consider the two-component Gaussian mixture

$$\frac{1}{2}\mathcal{N}(0, I_N) + \frac{1}{2}\mathcal{N}(0, 4\, I_N).$$

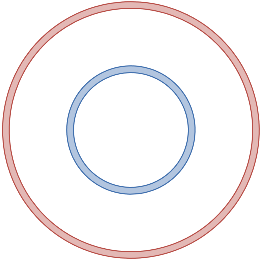
- Unimodal, but awful concentration.
- Without log-concavity, we lack tools to reason about such distributions.



Unfortunately, some unimodal distributions do not behave so nicely. Consider the two-component Gaussian mixture

$$\frac{1}{2}\mathcal{N}(0, I_N) + \frac{1}{2}\mathcal{N}(0, 4\, I_N).$$

- Unimodal, but awful concentration.
- Without log-concavity, we lack tools to reason about such distributions.



This talk will provide one such tool.

- Results show **confinement**, which is weaker than concentration.

Plan for the Talk

1. Ginzburg–Landau Surfaces and Main Results
2. Confinement from the Gaussian Correlation Inequality
3. The FKG-Gaussian Correlation Inequality
4. Putting it all together

Discrete Gaussian Free Fields

Warmup: Gaussian free field (GFF) on locally finite graph $G = (V, E)$:

- Fix finite $\Lambda \subseteq V$, e.g. $[-L, \ldots, L]^d \subseteq \mathbb{Z}^d$. Set $\varphi(v) = 0$ for all $v \notin \Lambda$.

Discrete Gaussian Free Fields

Warmup: Gaussian free field (GFF) on locally finite graph $G = (V, E)$:

- Fix finite $\Lambda \subseteq V$, e.g. $[-L, \dots, L]^d \subseteq \mathbb{Z}^d$. Set $\varphi(v) = 0$ for all $v \notin \Lambda$.
- GFF is the random function $\varphi : V \to \mathbb{R}$ with density:

$$d\mu_{G,\Lambda,GFF}(\varphi) = \frac{1}{Z_{G,\Lambda,GFF}} \exp\left(-\sum_{e=\{v,v'\}\in E} \frac{1}{2}|\varphi(v)-\varphi(v')|^2\right) \prod_{v\in\Lambda} d\varphi(v)$$

Discrete Gaussian Free Fields

Warmup: Gaussian free field (GFF) on locally finite graph $G = (V, E)$:

- Fix finite $\Lambda \subseteq V$, e.g. $[-L, \ldots, L]^d \subseteq \mathbb{Z}^d$. Set $\varphi(v) = 0$ for all $v \notin \Lambda$.
- GFF is the random function $\varphi : V \to \mathbb{R}$ with density:

$$d\mu_{G,\Lambda,GFF}(\varphi) = \frac{1}{Z_{G,\Lambda,GFF}} \exp\left(-\sum_{e=\{v,v'\}\in E} \frac{1}{2}|\varphi(v)-\varphi(v')|^2\right) \prod_{v\in\Lambda} d\varphi(v)$$

$$= \frac{1}{Z_{G,\Lambda,GFF}} \exp\left(-\sum_{e\in E} \frac{1}{2} \cdot |\nabla\varphi(e)|^2\right) \prod_{v\in\Lambda} d\varphi(v).$$

Discrete Gaussian Free Fields

Warmup: Gaussian free field (GFF) on locally finite graph $G = (V, E)$:

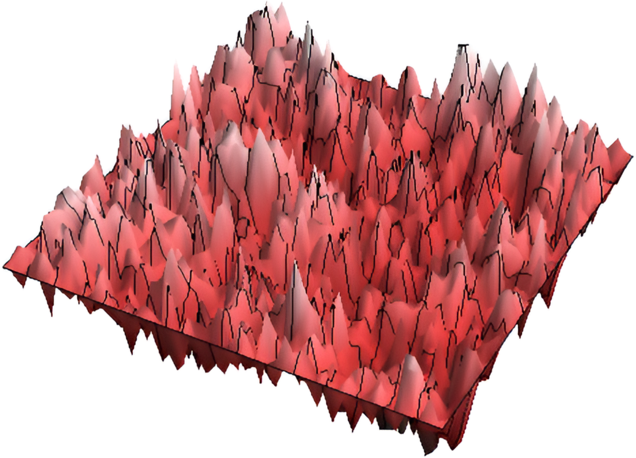
- Fix finite $\Lambda \subseteq V$, e.g. $[-L, \ldots, L]^d \subseteq \mathbb{Z}^d$. Set $\varphi(v) = 0$ for all $v \notin \Lambda$.
- GFF is the random function $\varphi : V \to \mathbb{R}$ with density:

$$d\mu_{G,\Lambda,GFF}(\varphi) = \frac{1}{Z_{G,\Lambda,GFF}} \exp\left(-\sum_{e=\{v,v'\}\in E} \frac{1}{2}|\varphi(v)-\varphi(v')|^2\right) \prod_{v\in\Lambda} d\varphi(v)$$

$$= \frac{1}{Z_{G,\Lambda,GFF}} \exp\left(-\sum_{e\in E} \frac{1}{2} \cdot |\nabla\varphi(e)|^2\right) \prod_{v\in\Lambda} d\varphi(v).$$

- Models fluctuations of random interfaces.
- Lots of probabilistic interest, notably on $\mathbb{Z}^2$ (extreme values, LQG).

Discrete Gaussian Free Fields



Picture of GFF by Sam Watson. Here $\Lambda = [-L, \ldots, L]^2 \subseteq \mathbb{Z}^2$.

Discrete Gaussian Free Fields and Electrical Networks

Well-known link between GFF and electrical networks:

- Let $R_{eff}(\cdot)$ be effective resistance on G. Then

$$\mathbb{E}^{\mu_{G,\Lambda,GFF}}[\varphi(v)^2] = R_{eff}(v \leftrightarrow \partial\Lambda).$$

- More generally, $\mathbb{E}[(\varphi(v)-\varphi(w))^2] = R_{eff}(v \leftrightarrow w)$.
- $R_{eff}(v \leftrightarrow \infty) < \infty$ iff simple random walk on G is transient.

Well-known link between GFF and electrical networks:

- Let $R_{eff}(\cdot)$ be effective resistance on G. Then

$$\mathbb{E}^{\mu_{G,\Lambda,GFF}}[\varphi(v)^2] = R_{eff}(v \leftrightarrow \partial\Lambda).$$

- More generally, $\mathbb{E}[(\varphi(v) - \varphi(w))^2] = R_{eff}(v \leftrightarrow w)$.
- $R_{eff}(v \leftrightarrow \infty) < \infty$ iff simple random walk on G is transient.

Definition of effective resistance: the **energy** of $f : E(G) \to \mathbb{R}$ is

$$\mathcal{E}(f) = \sum_{e \in E} f(e)^2.$$

$R_{eff}(v \leftrightarrow w)$ is the minimum energy of any $v \to w$ unit flow.

More general Ginzburg–Landau surfaces use non-quadratic interactions:

$$d\mu_{G,\Lambda,U}(\varphi) \equiv \frac{1}{Z_{G,\Lambda,U}} \exp\left(-\sum_{e\in E} U(\nabla\varphi(e)) \right) \prod_{v\in\Lambda} d\varphi(v).$$

We always assume $U : \mathbb{R} \to \mathbb{R}$ is even, i.e. $U(x) = U(-x)$.

More general Ginzburg–Landau surfaces use non-quadratic interactions:

$$d\mu_{G,\Lambda,U}(\varphi) \equiv \frac{1}{Z_{G,\Lambda,U}} \exp\left(-\sum_{e\in E} U(\nabla\varphi(e))\right) \prod_{v\in\Lambda} d\varphi(v).$$

We always assume $U:\mathbb{R}\to\mathbb{R}$ is even, i.e. $U(x) = U(-x)$.

- First rigorous study in [Brascamp-Lieb-Lebowitz 1975].
- Names: "Ginzburg–Landau", "$\nabla\varphi$", "anharmonic crystal".
- Dynamics, large deviations, fluctuations, $\mathbb{Z}$-valued analogs,... [Funaki-Spohn 97, Naddaf-Spencer 97, Deuschel-Giacomin-Ioffe 00, Sheffield 03, Miller 11, Armstrong-Dario 22, Armstrong-Wu 23...].

Localization

We will consider the question of **localization**.

Question

Are fluctuations of $\varphi(v_0)$ stochastically bounded on large domains $\Lambda \uparrow V$? If so, we say the model is **localized**. Otherwise **delocalized**.

Localization

We will consider the question of **localization**.

Question

Are fluctuations of $\varphi(v_0)$ stochastically bounded on large domains $\Lambda \uparrow V$? If so, we say the model is **localized**. Otherwise **delocalized**.

- Localization implies existence of infinite volume Gibbs measures.
  - One can even take this as the definition of localization.
- GFF on $\mathbb{Z}^d$ localizes iff $d \geq 3$.
  - Equivalent to transience/recurrence since

$$\mathbb{E}^{\mu_{G,\Lambda,GFF}}[\varphi(v)^2] = R_{eff}(v \leftrightarrow \partial\Lambda).$$

  - On $[-L, \ldots, L]^2 \subseteq \mathbb{Z}^2$, one has $\mathbb{E}[\varphi(\vec{0})^2] \approx \log L$.

Localization

We will consider the question of **localization**.

Question

Are fluctuations of $\varphi(v_0)$ stochastically bounded on large domains $\Lambda \uparrow V$? If so, we say the model is **localized**. Otherwise **delocalized**.

- Localization implies existence of infinite volume Gibbs measures.
  - One can even take this as the definition of localization.
- GFF on $\mathbb{Z}^d$ localizes iff $d \geq 3$.
  - Equivalent to transience/recurrence since

$$\mathbb{E}^{\mu_{G,\Lambda,GFF}}[\varphi(v)^2] = R_{eff}(v \leftrightarrow \partial\Lambda).$$

  - On $[-L,\dots,L]^2 \subseteq \mathbb{Z}^2$, one has $\mathbb{E}[\varphi(\vec{0})^2] \approx \log L$.

- Conjecture of [Brascamp-Lieb-Lebowitz 1975]: localization is determined by the **geometry** of G, not the potential U.
  - Proved delocalization for very general $U \in C^2(\mathbb{R})$ on $\mathbb{Z}^2$.

Localization of Ginzburg–Landau Random Surfaces

Localization is known for various U (often focused on lattices):

- Strongly convex potentials with $\inf_{x \in \mathbb{R}} U''(x) \geq c > 0$.
  - $\mu_{G,\Lambda,U}$ is dominated by GFF [Brascamp-Lieb-Lebowitz 75].

Localization is known for various U (often focused on lattices):

- Strongly convex potentials with $\inf_{x \in \mathbb{R}} U''(x) \geq c > 0$.
  - $\mu_{G,\Lambda,U}$ is dominated by GFF [Brascamp-Lieb-Lebowitz 75].
- $U(x) = |x|$ using infrared bounds [Bricmont-Fontaine-Lebowitz 82].
- **Mildly** non-convex U via renormalization [Cotar-Deuschel-Muller 09 & 12, ABKM 16 & 19, Hilger 16 & 20].

Localization is known for various U (often focused on lattices):

- Strongly convex potentials with $\inf_{x \in \mathbb{R}} U''(x) \geq c > 0$.
  - $\mu_{G,\Lambda,U}$ is dominated by GFF [Brascamp-Lieb-Lebowitz 75].
- $U(x) = |x|$ using infrared bounds [Bricmont-Fontaine-Lebowitz 82].
- **Mildly** non-convex U via renormalization [Cotar-Deuschel-Muller 09 & 12, ABKM 16 & 19, Hilger 16 & 20].
- $e^{-U(x)}$ is a mixture of centered Gaussians (will explain soon) [Biskup-Kotecky 07, Biskup-Spohn 11, Brydges-Spencer 12, Ye 19,...].
- [Magazinov-Peled 22]: convex U with $U''(x) > 0$ for a.e. x.

Localization of Ginzburg–Landau Random Surfaces

Localization is known for various U (often focused on lattices):

- Strongly convex potentials with $\inf_{x\in\mathbb{R}} U''(x) \geq c > 0$.
  - $\mu_{G,\Lambda,U}$ is dominated by GFF [Brascamp-Lieb-Lebowitz 75].

- $U(x) = |x|$ using infrared bounds [Bricmont-Fontaine-Lebowitz 82].
- **Mildly** non-convex U via renormalization [Cotar-Deuschel-Muller 09 & 12, ABKM 16 & 19, Hilger 16 & 20].

- $e^{-U(x)}$ is a mixture of centered Gaussians (will explain soon) [Biskup-Kotecky 07, Biskup-Spohn 11, Brydges-Spencer 12, Ye 19,...].
- [Magazinov-Peled 22]: convex U with $U''(x) > 0$ for a.e. x.

- Still open for **Hammock potential** $U(x) = \infty \cdot 1_{|x|>1}$. This gives a uniformly random 1-Lipschitz $\varphi : V \to \mathbb{R}$.

Main Result: Localization for Monotone Potentials

We prove localization for **monotone** potentials.

Definition ((α, ε)-monotonicity)

U is (α, ε)-monotone if it is even, increasing on $\mathbb{R}^+$, and $U'(x) \geq \min\left(\varepsilon x, \frac{1+\alpha}{x}\right)$ for all points of differentiability $x \geq 0$.

Main Result: Localization for Monotone Potentials

We prove localization for **monotone** potentials.

Definition ((α, ε)-monotonicity)

U is (α, ε)-monotone if it is even, increasing on $\mathbb{R}^+$, and $U'(x) \geq \min\left(\varepsilon x, \frac{1+\alpha}{x}\right)$ for all points of differentiability $x \geq 0$.

Theorem (Localization for (α, ε)-monotone U)

Let G be transient, and U be (α, ε)-monotone for $\alpha > 2$. Then $\mathbb{P}^{\mu_{G,\Lambda,U}}[|\varphi(v_0)| \geq t] \leq O(t^{-\alpha})$ uniformly in $\Lambda \subseteq V$, for any $v_0 \in V$.

Main Result: Localization for Monotone Potentials

We prove localization for **monotone** potentials.

Definition ((α, ε)-monotonicity)

U is (α, ε)-monotone if it is even, increasing on $\mathbb{R}^+$, and $U'(x) \geq \min\left(\varepsilon x, \frac{1+\alpha}{x}\right)$ for all points of differentiability $x \geq 0$.

Theorem (Localization for (α, ε)-monotone U)

Let G be transient, and U be (α, ε)-monotone for $\alpha > 2$. Then $\mathbb{P}^{\mu_{G,\Lambda,U}}[|\varphi(v_0)| \geq t] \leq O(t^{-\alpha})$ uniformly in $\Lambda \subseteq V$, for any $v_0 \in V$.

- Proof will be based on unimodality of $\mu_{G,\Lambda,U}$.
- U_e can depend on edge e, as long as (α, ε) are uniform.

Main Result: Localization for Monotone Potentials

We prove localization for **monotone** potentials.

Definition ((α, ε)-monotonicity)

U is (α, ε)-monotone if it is even, increasing on $\mathbb{R}^+$, and $U'(x) \geq \min\left(\varepsilon x, \frac{1+\alpha}{x}\right)$ for all points of differentiability $x \geq 0$.

Theorem (Localization for (α, ε)-monotone U)

Let G be transient, and U be (α, ε)-monotone for $\alpha > 2$. Then $\mathbb{P}^{\mu_{G,\Lambda,U}}[|\varphi(v_0)| \geq t] \leq O(t^{-\alpha})$ uniformly in $\Lambda \subseteq V$, for any $v_0 \in V$.

- Proof will be based on unimodality of $\mu_{G,\Lambda,U}$.
- U_e can depend on edge e, as long as (α, ε) are uniform.

- If G is transient and **transitive**, $\varphi(v_0)$ is tight even for $\alpha = \varepsilon$.
  - $\approx$ minimal condition for $\int_{\mathbb{R}} e^{-U(x)} dx < \infty$ so $Z_{G,\Lambda,U} < \infty$.

Extreme Values of the Field

These bounds are often sharp enough to understand $\max_{v\in\Lambda}|\varphi(v)|$.

Theorem (Extreme Values from Polynomial Bounds)

Let U be (α,ε)-monotone with $\sup_{x\geq 1}|U(x)-(1+\alpha)\log x|<\infty$ and $\alpha>2$. As $\Lambda\subseteq\mathbb{Z}^d$ varies for $d\geq 3$, the laws of

$$|\Lambda|^{-\frac{1}{2d\alpha}}\max_{v\in\Lambda}|\varphi(v)|$$

are tight in $(0,\infty)$, i.e. stochastically bounded away from 0 and ∞.

Extreme Values of the Field

These bounds are often sharp enough to understand $\max_{v\in\Lambda}|\varphi(v)|$.

Theorem (Extreme Values from Polynomial Bounds)

Let U be (α,ε)-monotone with $\sup_{x\geq 1}|U(x)-(1+\alpha)\log x|<\infty$ and $\alpha>2$. As $\Lambda\subseteq\mathbb{Z}^d$ varies for $d\geq 3$, the laws of

$$|\Lambda|^{-\frac{1}{2d\alpha}}\max_{v\in\Lambda}|\varphi(v)|$$

are tight in $(0,\infty)$, i.e. stochastically bounded away from 0 and ∞.

- Lower bound: condition outside a large independent set $\mathcal{I}\subseteq\Lambda$.
- Upper bound: Markov with extra tricks to get $2d\alpha$ in the exponent. (Split $\mathbb{Z}^d$ into $2d$ transient subgraphs containing the vertex v...)

Extreme Values of the Field

These bounds are often sharp enough to understand $\max_{v \in \Lambda} |\varphi(v)|$.

Theorem (Extreme Values from Polynomial Bounds)

Let U be (α, ε)-monotone with $\sup_{x \geq 1} |U(x) - (1+\alpha)\log x| < \infty$ and $\alpha > 2$. As $\Lambda \subseteq \mathbb{Z}^d$ varies for $d \geq 3$, the laws of

$$|\Lambda|^{-\frac{1}{2d\alpha}} \max_{v \in \Lambda} |\varphi(v)|$$

are tight in $(0, \infty)$, i.e. stochastically bounded away from 0 and ∞.

- Lower bound: condition outside a large independent set $\mathcal{I} \subseteq \Lambda$.
- Upper bound: Markov with extra tricks to get $2d\alpha$ in the exponent. (Split $\mathbb{Z}^d$ into $2d$ transient subgraphs containing the vertex v...)
- Similar condition for stretched exponential tails.
  - Monotonicity condition: $U'(x) \geq \min\left(\varepsilon x, \varepsilon x^{\beta-1}\right)$, for $\beta \in (0, 2]$.

General Statement without Graphs

The graph structure is irrelevant in the main result!

Let U be (α, ε)-monotone for $\alpha > 2$, and $\ell_1, \dots, \ell_j : \mathbb{R}^d \to \mathbb{R}$ be linear.

Choose $\varphi, \widetilde{\varphi} \in \mathbb{R}^d$ from densities:

$$\varphi \sim \exp\Big(\sum_{i=1}^{j} -U(\ell_i(\varphi))\Big) d\varphi / Z_{\vec{\ell},U},$$
$$\widetilde{\varphi} \sim \exp\Big(\sum_{i=1}^{j} -\ell_i(\widetilde{\varphi})^2\Big) d\widetilde{\varphi} / Z_{\vec{\ell},GFF}.$$

Fix any other linear function $\ell_* : \mathbb{R}^d \to \mathbb{R}$. Then $\ell_*(\varphi)$ is bounded **on the same scale** as the centered Gaussian $\ell_*(\widetilde{\varphi})$, with α-power tails.

- Recovering GFF/Ginzburg-Landau: set $\ell_e(\varphi) = \varphi(v) - \varphi(v')$.

Preview of the Proof

The proof has two core components:

1. Handle the case that $U = V$ takes the form:

$$e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi).$$

Preview of the Proof

The proof has two core components:

1. Handle the case that $U = V$ takes the form:

$$e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi).$$

- Each potential is a mixture of centered Gaussians.
- The overall model will be a mixture of Gaussian processes, e.g. GFFs with edge weights [Biskup-Kotecky 07].
- [Biskup-Spohn 11]: can have phase coexistence $\approx$ non-concentration.

Preview of the Proof

The proof has two core components:

1. Handle the case that $U = V$ takes the form:

$$e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi).$$

   - Each potential is a mixture of centered Gaussians.
   - The overall model will be a mixture of Gaussian processes, e.g. GFFs with edge weights [Biskup-Kotecky 07].
   - [Biskup-Spohn 11]: can have phase coexistence $\approx$ non-concentration.

2. Reduce to this case using the FKG-Gaussian correlation inequality. Amounts to **domination by mixtures of centered Gaussians**.
   - Dominating Gaussian mixtures must have special structure.
   - Perfectly suited for products of 1-dimensional functions.

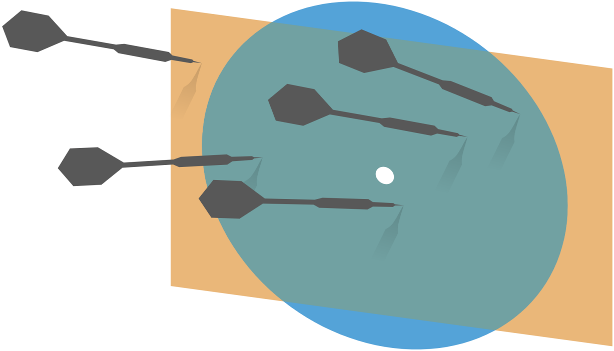
Plan for the Talk

1. Results on Ginzburg–Landau Model
2. Confinement from the Gaussian Correlation Inequality
3. The FKG-Gaussian Correlation Inequality
4. Putting it all together

Theorem (Royen 2014)

Let γ be a centered Gaussian measure on $\mathbb{R}^d$, and $K_1, K_2 \subseteq \mathbb{R}^d$ symmetric convex sets (i.e. $K_i = -K_i$). Then 1_{K_1} and 1_{K_2} have non-negative correlation under γ, i.e.

$$\gamma(K_1 \cap K_2) \geq \gamma(K_1)\gamma(K_2).$$



The Gaussian Correlation Inequality (GCI)

Theorem (Royen 2014)

For γ centered Gaussian on $\mathbb{R}^d$, and $K_1, K_2 \subseteq \mathbb{R}^d$ symmetric convex sets:

$$\gamma(K_1 \cap K_2) \geq \gamma(K_1)\gamma(K_2).$$

History (see 2017 Quanta article):

- Conjectured by [Dunnet-Sobel 55], [Gupta-Eaton-Perlman-Savage-Sobel 72].
- [Khatri 67, Sidak 67, Pitts 77, Schechtman-Schlumprecht-Zinn 98, Hargé 99]: special cases such as $\mathbb{R}^2$.
- **[Royen 2014]**: amazing solution. Initially escapes attention.
- [Latała-Matlak 2015]: exposition of Royen's proof

The Gaussian Correlation Inequality (GCI)

Theorem (Royen 2014)

For γ centered Gaussian on $\mathbb{R}^d$, and $K_1, K_2 \subseteq \mathbb{R}^d$ symmetric convex sets:

$$\gamma(K_1 \cap K_2) \geq \gamma(K_1)\gamma(K_2).$$

History (see 2017 Quanta article):

- Conjectured by [Dunnet-Sobel 55], [Gupta-Eaton-Perlman-Savage-Sobel 72].
- [Khatri 67, Sidak 67, Pitts 77, Schechtman-Schlumprecht-Zinn 98, Hargé 99]: special cases such as $\mathbb{R}^2$.
- **[Royen 2014]**: amazing solution. Initially escapes attention.
- [Latała-Matlak 2015]: exposition of Royen's proof

Proof idea: for $x, y \overset{i.i.d.}{\sim} \gamma$, equivalent to

$$\mathbb{P}[x \in K_1 \wedge x \in K_2] \geq \mathbb{P}[x \in K_1, y \in K_2].$$

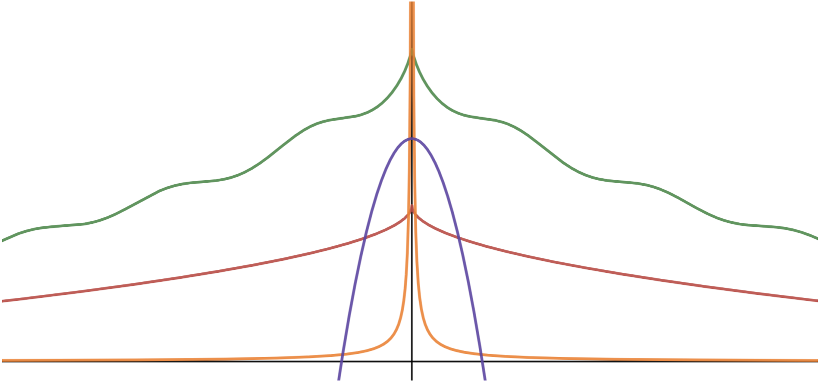
Royen showed $f(t) = \mathbb{P}[x \in K_1 \;\; \wedge \;\; \sqrt{1-t}x + \sqrt{t}y \in K_2]$ is decreasing.

Symmetric Quasi-Concave Functions

Definition

$f : \mathbb{R}^N \to \mathbb{R}$ is symmetric quasi-concave (SQC) if:

- $f(x) = f(-x)$ for all $x \in \mathbb{R}^N$.
- All super-level sets $\{x \in \mathbb{R}^N \; : \; f(x) \geq \lambda\}$ are convex.



GCI Yields Confinement

GCI: if $K_1, K_2 \subseteq \mathbb{R}^d$ are symmetric convex, then

$$\gamma(K_1 \cap K_2) \geq \gamma(K_1)\gamma(K_2).$$

GCI Yields Confinement

GCI: if $K_1, K_2 \subseteq \mathbb{R}^d$ are symmetric convex, then

$$\gamma(K_1 \cap K_2) \geq \gamma(K_1)\gamma(K_2).$$

If $K_1, \ldots, K_{m+1} \subseteq \mathbb{R}^d$ are symmetric convex:

$$\gamma(K_1 \cap \cdots \cap K_{m+1}) \geq \gamma(K_1 \cap \cdots \cap K_m) \cdot \gamma(K_{m+1}),$$

GCI Yields Confinement

GCI: if $K_1, K_2 \subseteq \mathbb{R}^d$ are symmetric convex, then

$$\gamma(K_1 \cap K_2) \geq \gamma(K_1)\gamma(K_2).$$

If $K_1, \ldots, K_{m+1} \subseteq \mathbb{R}^d$ are symmetric convex:

$$\gamma(K_1 \cap \cdots \cap K_{m+1}) \geq \gamma(K_1 \cap \cdots \cap K_m) \cdot \gamma(K_{m+1}),$$

By level sets, if $f_1, \ldots, f_{m+1} : \mathbb{R}^d \to \mathbb{R}^+$ are symmetric quasi-concave,

$$\mathbb{E}^\gamma[f_1 f_2 \ldots f_{m+1}] \geq \mathbb{E}^\gamma[f_1 f_2 \ldots f_m] \cdot \mathbb{E}^\gamma[f_{m+1}].$$

(Products of SQC functions need not be SQC, hence the middle step.)

GCI Yields Confinement

If $f_1, \ldots, f_{m+1} : \mathbb{R}^d \to \mathbb{R}^+$ are symmetric quasi-concave,

$$\mathbb{E}^{\gamma}[f_1 f_2 \ldots f_{m+1}] \geq \mathbb{E}^{\gamma}[f_1 f_2 \ldots f_m] \cdot \mathbb{E}^{\gamma}[f_{m+1}].$$

GCI Yields Confinement

If $f_1, \ldots, f_{m+1} : \mathbb{R}^d \to \mathbb{R}^+$ are symmetric quasi-concave,

$$\mathbb{E}^{\gamma}[f_1 f_2 \ldots f_{m+1}] \geq \mathbb{E}^{\gamma}[f_1 f_2 \ldots f_m] \cdot \mathbb{E}^{\gamma}[f_{m+1}].$$

Suppose γ is centered Gaussian and $\frac{d\nu}{d\gamma} = f_1 f_2 \ldots f_m$ is a product of SQC functions. Then

$$\nu(K) = \mathbb{E}^{\gamma}\left[\frac{d\nu}{d\gamma} \cdot 1_K\right] \overset{\text{GCI}}{\geq} \mathbb{E}^{\gamma}\left[\frac{d\nu}{d\gamma}\right] \cdot \gamma(K) = \gamma(K)$$

for symmetric convex K.

GCI Yields Confinement

If $f_1, \ldots, f_{m+1} : \mathbb{R}^d \to \mathbb{R}^+$ are symmetric quasi-concave,

$$\mathbb{E}^{\gamma}[f_1 f_2 \ldots f_{m+1}] \geq \mathbb{E}^{\gamma}[f_1 f_2 \ldots f_m] \cdot \mathbb{E}^{\gamma}[f_{m+1}].$$

Suppose γ is centered Gaussian and $\frac{d\nu}{d\gamma} = f_1 f_2 \ldots f_m$ is a product of SQC functions. Then

$$\nu(K) = \mathbb{E}^{\gamma}\left[\frac{d\nu}{d\gamma} \cdot 1_K\right] \overset{\text{GCI}}{\geq} \mathbb{E}^{\gamma}\left[\frac{d\nu}{d\gamma}\right] \cdot \gamma(K) = \gamma(K)$$

for symmetric convex K.

This is a form of **Gaussian domination**. We say $\nu \preceq_{\text{con}} \gamma$.

Definition

$\nu \preceq_{\text{con}} \gamma$ if $\gamma(K) \leq \nu(K)$ for all symmetric convex sets K.

Application: an Easier Case of Localization

Consequence: localization on all transient G if $U'(x) \geq \varepsilon x$ for all $x \geq 0$.

Why? Domination by rescaled GFF γ_ε with potential $U_\varepsilon(x) = \varepsilon x^2/2$.

Application: an Easier Case of Localization

Consequence: localization on all transient G if $U'(x) \geq \varepsilon x$ for all $x \geq 0$.

Why? Domination by rescaled GFF γ_ε with potential $U_\varepsilon(x) = \varepsilon x^2/2$.

The Radon–Nikodym derivative is a product:

$$\frac{d\mu_{G,\Lambda,U}}{d\gamma_\varepsilon} \propto \prod_{e \in E(G)} W(\nabla\varphi(e)), \quad \text{with} \quad W(x) = e^{-U(x)+U_\varepsilon(x)}.$$

Application: an Easier Case of Localization

Consequence: localization on all transient G if $U'(x) \geq \varepsilon x$ for all $x \geq 0$.

Why? Domination by rescaled GFF γ_ε with potential $U_\varepsilon(x) = \varepsilon x^2/2$.

The Radon–Nikodym derivative is a product:

$$\frac{d\mu_{G,\Lambda,U}}{d\gamma_\varepsilon} \propto \prod_{e \in E(G)} W(\nabla\varphi(e)), \quad \text{with} \quad W(x) = e^{-U(x)+U_\varepsilon(x)}.$$

Since $U' \geq U'_\varepsilon = \varepsilon x$, by assumption we find that W is SQC!

Application: an Easier Case of Localization

Consequence: localization on all transient G if $U'(x) \geq \varepsilon x$ for all $x \geq 0$.

Why? Domination by rescaled GFF γ_ε with potential $U_\varepsilon(x) = \varepsilon x^2/2$.

The Radon–Nikodym derivative is a product:

$$\frac{d\mu_{G,\Lambda,U}}{d\gamma_\varepsilon} \propto \prod_{e \in E(G)} W(\nabla\varphi(e)), \quad \text{with} \quad W(x) = e^{-U(x)+U_\varepsilon(x)}.$$

Since $U' \geq U'_\varepsilon = \varepsilon x$, by assumption we find that W is SQC!

Conclusion: $\mu_{G,\Lambda,U} \preceq_{\text{con}} \gamma_\varepsilon$. Localization on all transient G.

This method **requires** that $U(x) \geq \Omega(x^2)$. μ must have subgaussian tails to be dominated by a single Gaussian.

Plan for the Talk

1. Results on Ginzburg–Landau Model
2. Confinement from the Gaussian Correlation Inequality
3. **The FKG-Gaussian Correlation Inequality**
4. Putting it all together

GCI For Gaussian Mixtures

For heavy-tailed distributions, we cannot hope for Gaussian domination.

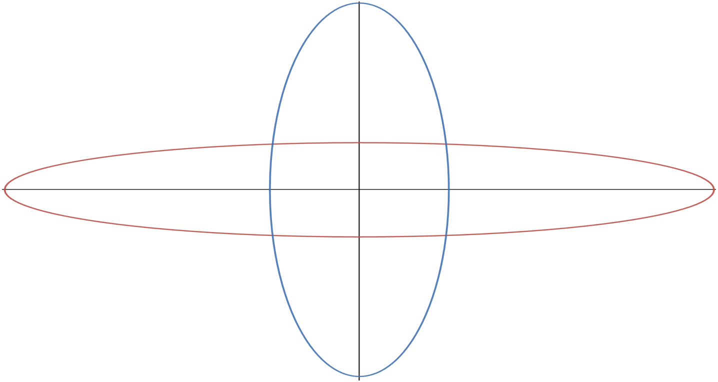
Does GCI extend to **mixtures** of centered Gaussians? If so, we could use them as the dominating measures and have more flexibility.

GCI For Gaussian Mixtures

For heavy-tailed distributions, we cannot hope for Gaussian domination.

Does GCI extend to **mixtures** of centered Gaussians? If so, we could use them as the dominating measures and have more flexibility.

- No, $\mu(K_1 \cap K_2) \geq \mu(K_1)\mu(K_2)$ is a non-linear condition in μ.
- Counterexample for two gaussians in $\mathbb{R}^2$:

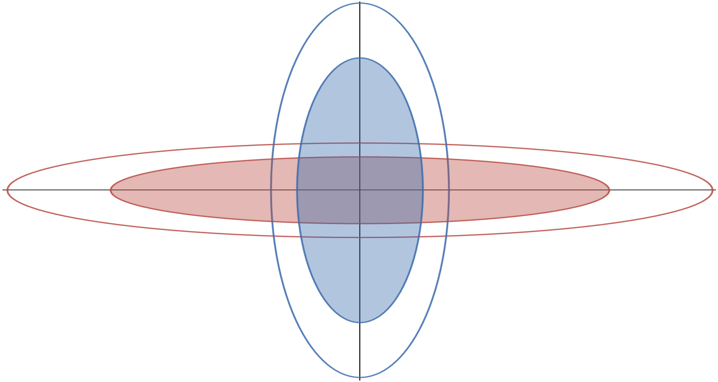


GCI For Gaussian Mixtures

For heavy-tailed distributions, we cannot hope for Gaussian domination.

Does GCI extend to **mixtures** of centered Gaussians? If so, we could use them as the dominating measures and have more flexibility.

- No, $\mu(K_1 \cap K_2) \geq \mu(K_1)\mu(K_2)$ is a non-linear condition in μ.
- Counterexample for two gaussians in $\mathbb{R}^2$:



GCI For Gaussian Mixtures

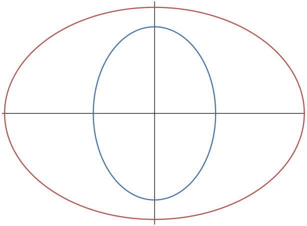
If the Gaussians have **comparable covariance**, GCI extends!!

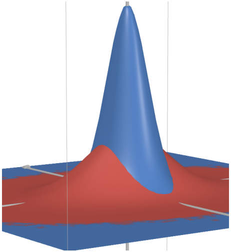
Theorem

Let $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} 0$ *be symmetric matrices. Let* $d\gamma_1(x) \propto e^{-\langle x, \Sigma_1 x\rangle}$ *and* $d\gamma_2(x) \propto e^{-\langle x, \Sigma_2 x\rangle}$. *Then GCI holds for* $\mu = p\gamma_1 + (1-p)\gamma_2$*:*

$$\mu(K \cap K') \geq \mu(K)\mu(K')$$

for any symmetric convex sets K, K' *and* $0 \leq p \leq 1$.





GCI For Gaussian Mixtures

Theorem

Let $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} 0$ *and* $\mathrm{d}\gamma_1(x) \propto e^{-\langle x, \Sigma_1 x \rangle}$ *and* $\mathrm{d}\gamma_2(x) \propto e^{-\langle x, \Sigma_2 x \rangle}$. *Then* $\mu(K \cap K') \geq \mu(K)\mu(K')$ *for* $\mu = p\gamma_1 + (1-p)\gamma_2$.

GCI For Gaussian Mixtures

Theorem

Let $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} 0$ *and* $d\gamma_1(x) \propto e^{-\langle x, \Sigma_1 x\rangle}$ *and* $d\gamma_2(x) \propto e^{-\langle x, \Sigma_2 x\rangle}$*.*
Then $\mu(K \cap K') \geq \mu(K)\mu(K')$ *for* $\mu = p\gamma_1 + (1-p)\gamma_2$*.*

Let $d\widetilde{\mu}(x) \propto 1_K d\mu(x)$. We will show $\widetilde{\mu}(K') \geq \mu(K')$.

GCI For Gaussian Mixtures

Theorem

Let $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} 0$ *and* $\mathrm{d}\gamma_1(x) \propto e^{-\langle x, \Sigma_1 x\rangle}$ *and* $\mathrm{d}\gamma_2(x) \propto e^{-\langle x, \Sigma_2 x\rangle}$. *Then* $\mu(K \cap K') \geq \mu(K)\mu(K')$ *for* $\mu = p\gamma_1 + (1-p)\gamma_2$.

Let $\mathrm{d}\widetilde{\mu}(x) \propto 1_K \mathrm{d}\mu(x)$. We will show $\widetilde{\mu}(K') \geq \mu(K')$.

- $\widetilde{\mu} = q\widetilde{\gamma}_1 + (1-q)\widetilde{\gamma}_2$, where $\mathrm{d}\widetilde{\gamma}_i(x) \propto 1_K \mathrm{d}\gamma_i(x)$ for $i \in \{1, 2\}$.

GCI For Gaussian Mixtures

Theorem

Let $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} 0$ *and* $d\gamma_1(x) \propto e^{-\langle x, \Sigma_1 x\rangle}$ *and* $d\gamma_2(x) \propto e^{-\langle x, \Sigma_2 x\rangle}$. *Then* $\mu(K \cap K') \geq \mu(K)\mu(K')$ *for* $\mu = p\gamma_1 + (1-p)\gamma_2$.

Let $d\widetilde{\mu}(x) \propto 1_K d\mu(x)$. We will show $\widetilde{\mu}(K') \geq \mu(K')$.

- $\widetilde{\mu} = q\widetilde{\gamma}_1 + (1-q)\widetilde{\gamma}_2$, where $d\widetilde{\gamma}_i(x) \propto 1_K d\gamma_i(x)$ for $i \in \{1, 2\}$.
- Can show $\gamma_1(K) \geq \gamma_2(K)$ (by GCI or otherwise). Thus $q \geq p$:

$$\frac{q(1-p)}{p(1-q)} = \frac{\gamma_1(K)}{\gamma_2(K)} \geq 1.$$

GCI For Gaussian Mixtures

Theorem

Let $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} 0$ and $d\gamma_1(x) \propto e^{-\langle x, \Sigma_1 x\rangle}$ and $d\gamma_2(x) \propto e^{-\langle x, \Sigma_2 x\rangle}$. Then $\mu(K \cap K') \geq \mu(K)\mu(K')$ for $\mu = p\gamma_1 + (1-p)\gamma_2$.

Let $d\widetilde{\mu}(x) \propto 1_K d\mu(x)$. We will show $\widetilde{\mu}(K') \geq \mu(K')$.

- $\widetilde{\mu} = q\widetilde{\gamma}_1 + (1-q)\widetilde{\gamma}_2$, where $d\widetilde{\gamma}_i(x) \propto 1_K d\gamma_i(x)$ for $i \in \{1, 2\}$.
- Can show $\gamma_1(K) \geq \gamma_2(K)$ (by GCI or otherwise). Thus $q \geq p$:

$$\frac{q(1-p)}{p(1-q)} = \frac{\gamma_1(K)}{\gamma_2(K)} \geq 1.$$

- Similarly $\gamma_1(K') \geq \gamma_2(K')$. **Finish by combining**.

GCI For Gaussian Mixtures

Theorem

Let $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} 0$ and $\mathrm{d}\gamma_1(x) \propto e^{-\langle x, \Sigma_1 x\rangle}$ and $\mathrm{d}\gamma_2(x) \propto e^{-\langle x, \Sigma_2 x\rangle}$. Then $\mu(K \cap K') \geq \mu(K)\mu(K')$ for $\mu = p\gamma_1 + (1-p)\gamma_2$.

Let $\mathrm{d}\widetilde{\mu}(x) \propto 1_K \mathrm{d}\mu(x)$. We will show $\widetilde{\mu}(K') \geq \mu(K')$.

- $\widetilde{\mu} = q\widetilde{\gamma}_1 + (1-q)\widetilde{\gamma}_2$, where $\mathrm{d}\widetilde{\gamma}_i(x) \propto 1_K \mathrm{d}\gamma_i(x)$ for $i \in \{1, 2\}$.
- Can show $\gamma_1(K) \geq \gamma_2(K)$ (by GCI or otherwise). Thus $q \geq p$:

$$\frac{q(1-p)}{p(1-q)} = \frac{\gamma_1(K)}{\gamma_2(K)} \geq 1.$$

- Similarly $\gamma_1(K') \geq \gamma_2(K')$. **Finish by combining.**

$$\widetilde{\mu}(K') = q\widetilde{\gamma}_1(K') + (1-q)\widetilde{\gamma}_2(K') \overset{GCI}{\geq} q\gamma_1(K') + (1-q)\gamma_2(K') \overset{Rearr}{\geq} p\gamma_1(K') + (1-p)\gamma_2(K') = \mu(K').$$

GCI For Totally Ordered Gaussian Mixtures

We can generalize further! Suppose:

1. $\mu = p_1\gamma_1 + \cdots + p_j\gamma_j$, with totally ordered inverse covariances $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} \cdots \succeq_{PSD} \Sigma_j$.
2. $d\widetilde{\mu}(x) \propto 1_K d\mu(x)$ for symmetric convex K.

GCI For Totally Ordered Gaussian Mixtures

We can generalize further! Suppose:

1. $\mu = p_1\gamma_1 + \cdots + p_j\gamma_j$, with totally ordered inverse covariances $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} \cdots \succeq_{PSD} \Sigma_j$.
2. $d\widetilde{\mu}(x) \propto 1_K d\mu(x)$ for symmetric convex K.

An analogous proof shows $\widetilde{\mu}(K') \geq \mu(K')$ for any symmetric convex K':

- $\widetilde{\mu} = q_1\widetilde{\gamma}_1 + \cdots + q_j\widetilde{\gamma}_j$.
- We similarly get:

$$\frac{q_1}{p_1} \geq \frac{q_2}{p_2} \geq \cdots \geq \frac{q_j}{p_j},$$
$$\gamma_1(K') \geq \gamma_2(K') \geq \cdots \geq \gamma_j(K').$$

GCI For Totally Ordered Gaussian Mixtures

We can generalize further! Suppose:

1. $\mu = p_1\gamma_1 + \cdots + p_j\gamma_j$, with totally ordered inverse covariances $\Sigma_1 \succeq_{PSD} \Sigma_2 \succeq_{PSD} \cdots \succeq_{PSD} \Sigma_j$.
2. $d\widetilde{\mu}(x) \propto 1_K d\mu(x)$ for symmetric convex K.

An analogous proof shows $\widetilde{\mu}(K') \geq \mu(K')$ for any symmetric convex K':

- $\widetilde{\mu} = q_1\widetilde{\gamma}_1 + \cdots + q_j\widetilde{\gamma}_j$.
- We similarly get:

$$\frac{q_1}{p_1} \geq \frac{q_2}{p_2} \geq \cdots \geq \frac{q_j}{p_j},$$

$$\gamma_1(K') \geq \gamma_2(K') \geq \cdots \geq \gamma_j(K').$$

We need these two functions on $\{1, 2, \ldots, j\}$ to be positively correlated with respect to the probability measure $\mathbb{P}[i] = p_i$.

This is the rearrangement inequality, **a special case of FKG**.

Log-Supermodular Gaussian Mixtures (LSGM)

Example (2×2 Lattice)

Let $p_{1,1}p_{2,2} \geq p_{1,2}p_{2,1}$. Suppose $d\gamma_{i,j}(x) \propto e^{-\langle x, \Sigma_{i,j} x \rangle}$ with:

$$\Sigma_{1,1} \succeq_{PSD} \Sigma_{1,2}, \Sigma_{2,1} \succeq_{PSD} \Sigma_{2,2} \succeq_{PSD} 0.$$

Then $\mu = p_{1,1}\gamma_{1,1} + p_{1,2}\gamma_{1,2} + p_{2,1}\gamma_{2,1} + p_{2,2}\gamma_{2,2}$ is an LSGM.

Log-Supermodular Gaussian Mixtures (LSGM)

Example (2×2 Lattice)

Let $p_{1,1}p_{2,2} \geq p_{1,2}p_{2,1}$. Suppose $d\gamma_{i,j}(x) \propto e^{-\langle x, \Sigma_{i,j} x \rangle}$ with:

$$\Sigma_{1,1} \succeq_{PSD} \Sigma_{1,2}, \Sigma_{2,1} \succeq_{PSD} \Sigma_{2,2} \succeq_{PSD} 0.$$

Then $\mu = p_{1,1}\gamma_{1,1} + p_{1,2}\gamma_{1,2} + p_{2,1}\gamma_{2,1} + p_{2,2}\gamma_{2,2}$ is an LSGM.

Definition (Log-Supermodular Gaussian Mixture)

An LSGM on $\mathbb{R}^n$ is a Gaussian mixture $\Gamma_{\nu,\Sigma} = \int \gamma_\xi \, d\nu(\xi)$. such that:

- $d\nu(x) = f(x)dx$ is log-supermodular on $\mathbb{R}_+^k$:

$$f(\xi)f(\xi') \leq f(\xi \wedge \xi')f(\xi \vee \xi'), \quad \forall \xi, \xi' \in \mathbb{R}_+^k.$$

- $d\gamma_\xi(x) \propto e^{-\langle x, \Sigma(\xi) x \rangle}$, for some $\Sigma : \mathbb{R}_+^k \to \mathcal{S}_+^n$.
- Σ is order-reversing from $\preceq_{\text{coord}}$ to $\preceq_{\text{PSD}}$.

The FKG-Gaussian Correlation Inequality

Theorem (FKG-GCI)

For any LSGM $\Gamma_{\nu,\Sigma}$ *and symmetric convex* K_1, K_2*:*

$$\Gamma_{\nu,\Sigma}(K_1 \cap K_2) \geq \Gamma_{\nu,\Sigma}(K_1)\Gamma_{\nu,\Sigma}(K_2).$$

The FKG-Gaussian Correlation Inequality

Theorem (FKG-GCI)

For any LSGM $\Gamma_{\nu,\Sigma}$ *and symmetric convex* K_1, K_2*:*

$$\Gamma_{\nu,\Sigma}(K_1 \cap K_2) \geq \Gamma_{\nu,\Sigma}(K_1)\Gamma_{\nu,\Sigma}(K_2).$$

Proof: One now needs increasing functions to be positively correlated w.r.t. the log-supermodular ν. This is the statement of FKG. □

The FKG-Gaussian Correlation Inequality

Theorem (FKG-GCI)

For any LSGM $\Gamma_{\nu,\Sigma}$ and symmetric convex K_1, K_2:

$$\Gamma_{\nu,\Sigma}(K_1 \cap K_2) \geq \Gamma_{\nu,\Sigma}(K_1)\Gamma_{\nu,\Sigma}(K_2).$$

Proof: One now needs increasing functions to be positively correlated w.r.t. the log-supermodular ν. This is the statement of FKG. □

Corollary (Domination by LSGM Gaussian Mixtures)

For any LSGM $\Gamma_{\nu,\Sigma}$, suppose for symmetric quasi-concave $f_1, \ldots, f_m$:

$$\mathrm{d}\widetilde{\Gamma}(x) = f_1(x) f_2(x) \ldots f_m(x)\, \mathrm{d}\Gamma_{\nu,\Sigma}(x).$$

Then $\widetilde{\Gamma} \preceq_{con} \Gamma_{\nu,\Sigma}$, i.e. $\widetilde{\Gamma}(K) \geq \Gamma_{\nu,\Sigma}(K)$ for all symmetric convex K.

The FKG-Gaussian Correlation Inequality

Theorem (FKG-GCI)

For any LSGM $\Gamma_{\nu,\Sigma}$ and symmetric convex K_1, K_2:
$$\Gamma_{\nu,\Sigma}(K_1 \cap K_2) \geq \Gamma_{\nu,\Sigma}(K_1)\Gamma_{\nu,\Sigma}(K_2).$$

Proof: One now needs increasing functions to be positively correlated w.r.t. the log-supermodular ν. This is the statement of FKG. □

Corollary (Domination by LSGM Gaussian Mixtures)

For any LSGM $\Gamma_{\nu,\Sigma}$, suppose for symmetric quasi-concave $f_1, \dots, f_m$:

$$d\widetilde{\Gamma}(x) = f_1(x) f_2(x) \dots f_m(x) \; d\Gamma_{\nu,\Sigma}(x).$$

Then $\widetilde{\Gamma} \preceq_{con} \Gamma_{\nu,\Sigma}$, i.e. $\widetilde{\Gamma}(K) \geq \Gamma_{\nu,\Sigma}(K)$ for all symmetric convex K.

Proof: Exactly the same level sets argument as from ordinary GCI. □

The FKG-Gaussian Correlation Inequality

Theorem (FKG-GCI)

For any LSGM $\Gamma_{\nu,\Sigma}$ and symmetric convex K_1, K_2:

$$\Gamma_{\nu,\Sigma}(K_1 \cap K_2) \geq \Gamma_{\nu,\Sigma}(K_1)\Gamma_{\nu,\Sigma}(K_2).$$

Proof: One now needs increasing functions to be positively correlated w.r.t. the log-supermodular ν. This is the statement of FKG. □

Corollary (Domination by LSGM Gaussian Mixtures)

For any LSGM $\Gamma_{\nu,\Sigma}$, suppose for symmetric quasi-concave $f_1, \ldots, f_m$:

$$d\widetilde{\Gamma}(x) = f_1(x)f_2(x)\ldots f_m(x)\ d\Gamma_{\nu,\Sigma}(x).$$

Then $\widetilde{\Gamma} \preceq_{con} \Gamma_{\nu,\Sigma}$, i.e. $\widetilde{\Gamma}(K) \geq \Gamma_{\nu,\Sigma}(K)$ for all symmetric convex K.

Proof: Exactly the same level sets argument as from ordinary GCI. □

For Ginzburg-Landau, it remains to express $\mu_{G,\Lambda,U}$ as $\widetilde{\Gamma}$ above.

Aside: Avoiding the Full Power of GCI

Although GCI is quite elegant, the proof is miraculous and took decades to find. For the purposes of this talk, it can be bypassed.

Aside: Avoiding the Full Power of GCI

Although GCI is quite elegant, the proof is miraculous and took decades to find. For the purposes of this talk, it can be bypassed.

"Slab GCI" (Sidak 68): let K, K' be symmetric convex sets with

$$K = \{v : |\langle v, x\rangle| \leq C\}$$

a **symmetric slab**. Then for centered Gaussian γ,

$$\gamma(K \cap K') \geq \gamma(K)\gamma(K').$$

Aside: Avoiding the Full Power of GCI

Although GCI is quite elegant, the proof is miraculous and took decades to find. For the purposes of this talk, it can be bypassed.

"Slab GCI" (Sidak 68): let K, K' be symmetric convex sets with

$$K = \{v : |\langle v, x\rangle| \leq C\}$$

a **symmetric slab**. Then for centered Gaussian γ,

$$\gamma(K \cap K') \geq \gamma(K)\gamma(K').$$

Proof idea: $\gamma|_K$ is strongly log-concave, so its projection onto x is too. $\square$

Aside: Avoiding the Full Power of GCI

Although GCI is quite elegant, the proof is miraculous and took decades to find. For the purposes of this talk, it can be bypassed.

"Slab GCI" (Sidak 68): let K, K' be symmetric convex sets with

$$K = \{v : |\langle v, x\rangle| \leq C\}$$

a **symmetric slab**. Then for centered Gaussian γ,

$$\gamma(K \cap K') \geq \gamma(K)\gamma(K').$$

Proof idea: $\gamma|_K$ is strongly log-concave, so its projection onto x is too. □

Instead of $\preceq_{\text{con}}$, one may consider the weaker relation

$$\nu \preceq_{slab} \mu$$

if $\nu(K) \geq \mu(K)$ for all symmetric slabs K. Then slab GCI yields a slab-FKG-GCI inequality. This will recover all our localization results, since they concern 1-dimensional projections of φ.

Plan for the Talk

1. Ginzburg–Landau Surfaces and Main Results
2. Confinement from the Gaussian Correlation Inequality
3. The FKG-Gaussian Correlation Inequality
4. **Putting it all together**

Log-Supermodular Gaussian Mixtures from Ginzburg-Landau

Dominating LSGMs will be $\mu_{G,\Lambda,V}$ where V takes the form:

$$e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi).$$

Log-Supermodular Gaussian Mixtures from Ginzburg-Landau

Dominating LSGMs will be $\mu_{G,\Lambda,V}$ where V takes the form:

$$e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi).$$

The Gibbs measure is a mixture of GFFs with random resistances ξ_e.

$$d\mu_{G,\vec{\xi},GFF}(\varphi) = \frac{1}{Z_{G,GFF}} \exp\left(-\sum_{e\in E} \frac{1}{2\xi_e{}^2} \cdot |\nabla\varphi(e)|^2 \right) \prod_{v\in V} d\varphi(v).$$

Dominating LSGMs will be $\mu_{G,\Lambda,V}$ where V takes the form:

$$e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi).$$

The Gibbs measure is a mixture of GFFs with random resistances ξ_e.

$$d\mu_{G,\vec{\xi},GFF}(\varphi) = \frac{1}{Z_{G,GFF}} \exp\left(-\sum_{e\in E} \frac{1}{2\xi_e{}^2} \cdot |\nabla\varphi(e)|^2 \right) \prod_{v\in V} d\varphi(v).$$

Encoding: inverse covariance $\Sigma(\vec{\xi})$ given by

$$\left\langle \varphi, \Sigma(\vec{\xi})\varphi \right\rangle = \sum_{e\in E(G)} (\varphi(v) - \varphi(v'))^2/\xi_e{}^2.$$

- Clearly Σ is order-reversing from $\preceq_{coord}$ to $\preceq_{PSD}$.

Log-Supermodular Gaussian Mixtures from Ginzburg-Landau

Potential $e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi)$. Encoding: inverse covariance

$$\left\langle \varphi, \Sigma(\vec{\xi})\varphi \right\rangle = \sum_{e \in E(G)} (\varphi(v) - \varphi(v'))^2 / \xi_e{}^2.$$

The Ginzburg-Landau measure is a mixture of these GFFs:

$$\mu_{G,\Lambda,V} = \int \mu_{G,\vec{\xi},GFF} \, d\nu(\vec{\xi}).$$

Log-Supermodular Gaussian Mixtures from Ginzburg-Landau

Potential $e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi)$. Encoding: inverse covariance

$$\left\langle \varphi, \Sigma(\vec{\xi})\varphi \right\rangle = \sum_{e \in E(G)} (\varphi(v) - \varphi(v'))^2 / {\xi_e}^2.$$

The Ginzburg-Landau measure is a mixture of these GFFs:

$$\mu_{G,\Lambda,V} = \int \mu_{G,\vec{\xi},GFF} \, d\nu(\vec{\xi}).$$

Mixing measure $d\nu(\vec{\xi})$ is **not** a product. It gains a factor $\det(\Sigma(\xi))^{-1/2}$.

Log-Supermodular Gaussian Mixtures from Ginzburg-Landau

Potential $e^{-V(x)} = \int_0^\infty \frac{e^{-x^2/2\xi^2}}{\xi\sqrt{2\pi}} d\rho(\xi)$. Encoding: inverse covariance

$$\left\langle \varphi, \Sigma(\vec{\xi})\varphi \right\rangle = \sum_{e \in E(G)} (\varphi(v) - \varphi(v'))^2 / \xi_e{}^2.$$

The Ginzburg-Landau measure is a mixture of these GFFs:

$$\mu_{G,\Lambda,V} = \int \mu_{G,\vec{\xi},GFF} \, d\nu(\vec{\xi}).$$

Mixing measure $d\nu(\vec{\xi})$ is **not** a product. It gains a factor $\det(\Sigma(\xi))^{-1/2}$.

- Elementary fact: if $A, B, C \succeq_{PSD} 0$, then

$$\det(A)\det(A+B+C) \leq \det(A+B)\det(A+C).$$

- This yields log-supermodularity, since $\Sigma(\vec{\xi}) = \sum_{e \in E} F(\xi_e)$ is additive.

Dominating Monotone Potentials by Gaussian Mixtures

If $U'(x) \geq V'(x)$ on $\mathbb{R}_+$, the Radon–Nikodym derivative

$$\frac{d\mu_{G,\Lambda,U}}{d\mu_{G,\Lambda,V}} \propto \prod_{e \in E} e^{-U(\nabla\varphi(e))+V(\nabla\varphi(e))}$$

is a product of symmetric quasi-concave functions.

Dominating Monotone Potentials by Gaussian Mixtures

If $U'(x) \geq V'(x)$ on $\mathbb{R}_+$, the Radon–Nikodym derivative

$$\frac{d\mu_{G,\Lambda,U}}{d\mu_{G,\Lambda,V}} \propto \prod_{e \in E} e^{-U(\nabla\varphi(e))+V(\nabla\varphi(e))}$$

is a product of symmetric quasi-concave functions. By FKG-GCI,

$$\mu_{G,\Lambda,U} \preceq_{\text{con}} \mu_{G,\Lambda,V}.$$

Dominating Monotone Potentials by Gaussian Mixtures

If $U'(x) \geq V'(x)$ on $\mathbb{R}_+$, the Radon–Nikodym derivative

$$\frac{d\mu_{G,\Lambda,U}}{d\mu_{G,\Lambda,V}} \propto \prod_{e \in E} e^{-U(\nabla\varphi(e))+V(\nabla\varphi(e))}$$

is a product of symmetric quasi-concave functions. By FKG-GCI,

$$\mu_{G,\Lambda,U} \preceq_{\text{con}} \mu_{G,\Lambda,V}.$$

Further, $\mu_{G,\Lambda,V}$ is dominated by the "naive product" LSGM:

$$\mu_{G,\Lambda,V} \preceq_{\text{con}} \int \mu_{G,\vec{\xi},GFF} \prod_{e \in E} d\rho(\xi_e).$$

Indeed the presence of $G \backslash \{e\}$ only makes ξ_e smaller, so one gets stochastic domination of $\vec{\xi}$ by $\prod_{e \in E} \rho(\xi_e)$. **This reduces localization to studying GFFs with IID edge resistances.**

Dominating Monotone Potentials by Gaussian Mixtures

Each monotonicity condition on U corresponds to an explicit $V = V(\rho)$.

Dominating Monotone Potentials by Gaussian Mixtures

Each monotonicity condition on U corresponds to an explicit $V = V(\rho)$.

Lemma

There exist potentials $V(\rho)$ in centered Gaussian mixture form such that:

1. $V'(x) \le \min\left(\varepsilon x, \frac{1+\varepsilon}{x}\right), \quad \forall x \ge 0.$
2. $V'(x) \le \min\left(\varepsilon x, \frac{1+\alpha}{x}\right)$ *and* $\rho([t,\infty)) \le O(t^{-\alpha}), \quad \forall t \ge 0.$
3. $V'(x) \le \min\left(\varepsilon x, \varepsilon x^{\beta-1}\right)$ *and* $\rho([t,\infty)) \le e^{-\Omega(t^{\beta})}, \quad \forall t \ge 0.$

In each case, $U' \ge V'$ if U is correspondingly monotone.

Dominating Monotone Potentials by Gaussian Mixtures

Each monotonicity condition on U corresponds to an explicit $V = V(\rho)$.

Lemma

There exist potentials $V(\rho)$ in centered Gaussian mixture form such that:

1. $V'(x) \leq \min\left(\varepsilon x, \frac{1+\varepsilon}{x}\right), \quad \forall x \geq 0.$
2. $V'(x) \leq \min\left(\varepsilon x, \frac{1+\alpha}{x}\right)$ *and* $\rho([t,\infty)) \leq O(t^{-\alpha}), \quad \forall t \geq 0.$
3. $V'(x) \leq \min\left(\varepsilon x, \varepsilon x^{\beta-1}\right)$ *and* $\rho([t,\infty)) \leq e^{-\Omega(t^{\beta})}, \quad \forall t \geq 0.$

In each case, $U' \geq V'$ if U is correspondingly monotone.

Proof Idea: Explicit construction. Match tail of ρ to the decay rate.

1. $\rho([t,\infty)) \asymp t^{-\varepsilon}$.
2. $\rho([t,\infty)) \asymp t^{-\alpha}$.
3. $\rho([t,\infty)) \asymp e^{-t^{\beta}}$. $\square$

Proof of Main Localization Result

Theorem

Fix $\alpha > 2$ *and transient* G. *Let* U *be* (α, ε)*-monotone. Then*

$$\mathbb{P}^{\mu_{G,\Lambda,U}}[|\varphi(v)| \geq t] \leq O(t^{-\alpha})$$

holds uniformly in Λ *for any transient* G.

Proof of Main Localization Result

Theorem

Fix $\alpha > 2$ *and transient* G. *Let* U *be* (α, ε)*-monotone. Then*

$$\mathbb{P}^{\mu_{G,\Lambda,U}}[|\varphi(v)| \geq t] \leq O(t^{-\alpha})$$

holds uniformly in Λ *for any transient* G.

We'll compare to GFF with IID $\xi_e \sim \rho$, with $\rho([t, \infty)) \leq O(t^{-\alpha})$.

Proof of Main Localization Result

Theorem

Fix $\alpha > 2$ and transient G. Let U be (α, ε)-monotone. Then

$$\mathbb{P}^{\mu_{G,\Lambda,U}}[|\varphi(v)| \geq t] \leq O(t^{-\alpha})$$

holds uniformly in Λ for any transient G.

We'll compare to GFF with IID $\xi_e \sim \rho$, with $\rho([t, \infty)) \leq O(t^{-\alpha})$.

Proof: The variance of $\varphi(v)$ after $\vec{\xi}$-weighting is a weighted effective resistance $R_{eff}^{(\xi)}(v \leftrightarrow \infty)$. So we must bound the tail of $R_{eff}^{(\xi)}(v \leftrightarrow \infty)$.

Proof of Main Localization Result

Theorem

Fix $\alpha > 2$ and transient G. Let U be (α, ε)-monotone. Then

$$\mathbb{P}^{\mu_{G,\Lambda,U}}[|\varphi(v)| \geq t] \leq O(t^{-\alpha})$$

holds uniformly in Λ for any transient G.

We'll compare to GFF with IID $\xi_e \sim \rho$, with $\rho([t, \infty)) \leq O(t^{-\alpha})$.

Proof: The variance of $\varphi(v)$ after $\vec{\xi}$-weighting is a weighted effective resistance $R_{eff}^{(\xi)}(v \leftrightarrow \infty)$. So we must bound the tail of $R_{eff}^{(\xi)}(v \leftrightarrow \infty)$.

Consider the energy-minimizing unit flow $v \to \infty$ in the **unweighted** G. Its **weighted** energy is random and upper bounds $R_{eff}^{(\xi)}(v \leftrightarrow \infty)$. This is $\sum_e a_e \xi_e^2$, where $\sum_e a_e = R_{eff}(v \leftrightarrow \infty) < \infty$. Now use e.g. Jensen. $\square$

Tightness from ε-Monotonicity

Theorem

Suppose U is $(\varepsilon, \varepsilon)$-monotone, and p-bond percolation on G has transient infinite cluster for $p \in [1-\delta, 1]$. Then Law$(\varphi(v))$ is tight as $\Lambda \uparrow \infty$.

(The condition holds for all transient transitive G by [Hutchcroft 23].)

Tightness from ε-Monotonicity

Theorem

Suppose U is $(\varepsilon, \varepsilon)$-monotone, and p-bond percolation on G has transient infinite cluster for $p \in [1-\delta, 1]$. Then Law$(\varphi(v))$ is tight as $\Lambda \uparrow \infty$.

(The condition holds for all transient transitive G by [Hutchcroft 23].)
Compare to GFF with IID edge resistances ξ_e, now with no tail bounds.

Tightness from ε-Monotonicity

Theorem

Suppose U is $(\varepsilon, \varepsilon)$-monotone, and p-bond percolation on G has transient infinite cluster for $p \in [1-\delta, 1]$. Then Law$(\varphi(v))$ is tight as $\Lambda \uparrow \infty$.

(The condition holds for all transient transitive G by [Hutchcroft 23].)
Compare to GFF with IID edge resistances ξ_e, now with no tail bounds.

Proof: take $\xi_e \sim \rho$ independent. Consider edges with $\xi_e \leq M$, where

$$\rho([0, M]) \geq 1 - \delta.$$

By definition, these edges form a transient infinite cluster $\mathcal{C}$.

Tightness from ε-Monotonicity

Theorem

Suppose U is $(\varepsilon, \varepsilon)$-monotone, and p-bond percolation on G has transient infinite cluster for $p \in [1-\delta, 1]$. Then $Law(\varphi(v))$ is tight as $\Lambda \uparrow \infty$.

(The condition holds for all transient transitive G by [Hutchcroft 23].)
Compare to GFF with IID edge resistances ξ_e, now with no tail bounds.

Proof: take $\xi_e \sim \rho$ independent. Consider edges with $\xi_e \le M$, where

$$\rho([0, M]) \ge 1 - \delta.$$

By definition, these edges form a transient infinite cluster $\mathcal{C}$.

Let $w \in \mathcal{C}$ be the closest point to v. Then both $R_{eff}^{(\xi)}(v \leftrightarrow w)$ and $R_{eff}^{(\xi)}(w \leftrightarrow \partial\Lambda)$ are tight. Hence $R_{eff}^{(\xi)}(v \leftrightarrow \partial\Lambda)$ is also tight. □

A Remark on Monotonicity

We used that if $U'(x) \geq V'(x)$ for all $x \geq 0$, then

$$\mu_{G,\Lambda,U} \preceq_{\text{con}} \mu_{G,\Lambda,V}.$$

This holds edge-by-edge. Hence if $\beta_e \geq 1$ for each $e \in E(G)$, then

$$\mu_{G,\Lambda,\beta_e V} \preceq_{\text{con}} \mu_{G,\Lambda,V}.$$

A Remark on Monotonicity

We used that if $U'(x) \geq V'(x)$ for all $x \geq 0$, then

$$\mu_{G,\Lambda,U} \preceq_{\text{con}} \mu_{G,\Lambda,V}.$$

This holds edge-by-edge. Hence if $\beta_e \geq 1$ for each $e \in E(G)$, then

$$\mu_{G,\Lambda,\beta_e V} \preceq_{\text{con}} \mu_{G,\Lambda,V}.$$

I.e. for Gaussian-mixture potentials V, "stiffening the springs" improves confinement. [BLL75] gives a **counterexample** for

$$V(x) = x^2 + \varepsilon x^4.$$

IV. STIFFENING THE SPRINGS DOES NOT NECESSARILY DECREASE $<x_0^2>$

Consider the following case with three particles, i.e. $H = v(x)+v(x-y)+v(y-z)+v(z)+\alpha v(x-z)$ and $x_0 = y$. Let $v(x) = x^2+\varepsilon x^4$, $\varepsilon > 0$. We want to show that increasing α from 0 can decrease $<x_0^2>$. Let g_∞ (resp. g_0) be $<x_0^2>$ for $\alpha = \infty$ (resp. $\alpha = 0$). Then $g_\infty = 2 \int y^2 G(y)/\int G(y)$ and $g_0 = \int y^2 F(y)/\int F(y)$, with $G(y) = \exp(-2v(y))$ and $F(y) = R(y)^2$ where $R = \exp(-v) * \exp(-v)$. A simple calculation shows that for the pure harmonic case ($\varepsilon = 0$), $g_0 = g_\infty = 1/2$. When $\varepsilon > 0$ it is impossible to calculate the integrals, but it is possible to calculate $g_i = dg_i/d\varepsilon|_{\varepsilon=0}$. One finds that $g_\infty = -3/4$ and $g_0 = -9/8$. Thus, for small, positive ε, $g_\infty > g_0$, which is the contradiction we wished to demonstrate.

Plan for the Talk

1. Ginzburg–Landau Surfaces and Main Results
2. Confinement from the Gaussian Correlation Inequality
3. The FKG-Gaussian Correlation Inequality
4. Putting it all together
5. **Another Application: the Fröhlich Polaron**

The Fröhlich Polaron

Let $d\mathbb{Q}(B)$ be the law of 3-dimensional Brownian motion.

Let $\mathrm{d}\mathbb{Q}(\mathrm{B})$ be the law of 3-dimensional Brownian motion.

Given coupling strength $\alpha \gg 1$ and time-horizon $T \gg \alpha$, the **Polaron path measure** $\widehat{\mathbb{Q}}_{\alpha,T}$ is the reweighted law on paths $B : [0,T] \to \mathbb{R}^3$:

$$\mathrm{d}\widehat{\mathbb{Q}}_{\alpha,T}(\mathrm{B}) \equiv \frac{1}{Z_{\alpha,T}} \exp\left(\alpha \int_0^T \int_0^T \frac{e^{-|t-s|}}{\|\mathrm{B}_t - \mathrm{B}_s\|} \, \mathrm{d}t \, \mathrm{d}s \right) \mathrm{d}\mathbb{Q}(\mathrm{B}).$$

The Fröhlich Polaron

Let $d\mathbb{Q}(B)$ be the law of 3-dimensional Brownian motion.

Given coupling strength $\alpha \gg 1$ and time-horizon $T \gg \alpha$, the **Polaron path measure** $\widehat{\mathbb{Q}}_{\alpha,T}$ is the reweighted law on paths $B : [0, T] \to \mathbb{R}^3$:

$$d\widehat{\mathbb{Q}}_{\alpha,T}(B) \equiv \frac{1}{Z_{\alpha,T}} \exp\left(\alpha \int_0^T \int_0^T \frac{e^{-|t-s|}}{\|B_t - B_s\|} \, dt \, ds\right) d\mathbb{Q}(B).$$

Obtained by Feynman's path integral applied to a quantum operator (modeling an electron in crystal). The "effective mass" is

$$m_{eff}(\alpha) = \lim_{T \to \infty} \frac{3T}{\mathbb{E}^{\widehat{\mathbb{Q}}_{\alpha,T}} \|B_T\|^2} \overset{?}{\approx} C_* \alpha^4.$$

[Fröhlich 37, Landau-Pekar 48, Feynman 55, Lieb 77, Donsker-Varadhan 83, Spohn 87, Lieb-Thomas 97, Lieb-Seiringer 17, Mukherjee-Varadhan 18 & 20, Dybalski-Spohn 20, Betz-Polzer 21 & 22, Brooks-Seiringer 22, **S** 22]

$$\mathrm{d}\widehat{\mathbb{Q}}_{\alpha,T}(\mathrm{B}) \equiv \frac{1}{Z_{\alpha,T}} \exp\left(\alpha \int_0^T \int_0^T \frac{e^{-|t-s|}}{\|\mathrm{B}_t - \mathrm{B}_s\|} \, \mathrm{d}t \, \mathrm{d}s \right) \mathrm{d}\mathbb{Q}(\mathrm{B}).$$

Wiener measure $\mathbb{Q}$ is Gaussian. $B_{[0,T]} \mapsto \frac{e^{-|t-s|}}{\|\mathrm{B}_t - \mathrm{B}_s\|}$ is SQC for any (s,t).

$$d\widehat{\mathbb{Q}}_{\alpha,T}(B) \equiv \frac{1}{Z_{\alpha,T}} \exp\left(\alpha \int_0^T \int_0^T \frac{e^{-|t-s|}}{\|B_t - B_s\|}\, dt\, ds\right) d\mathbb{Q}(B).$$

Wiener measure $\mathbb{Q}$ is Gaussian. $B_{[0,T]} \mapsto \frac{e^{-|t-s|}}{\|B_t - B_s\|}$ is SQC for any (s,t).

In fact, the Coulomb interaction is a mixture of centered Gaussians:

$$\frac{1}{x} = \sqrt{2/\pi} \int_0^\infty e^{-u^2x^2/2} du.$$

Hence $\widehat{\mathbb{Q}}_{\alpha,T}$ is an (infinite dimensional) LSGM and obeys GCI.

$$\mathrm{d}\widehat{\mathbb{Q}}_{\alpha,T}(\mathrm{B}) \equiv \frac{1}{Z_{\alpha,T}} \exp\left(\alpha \int_0^T \int_0^T \frac{e^{-|t-s|}}{\|\mathrm{B}_t - \mathrm{B}_s\|}\,\mathrm{d}t\,\mathrm{d}s\right) \mathrm{d}\mathbb{Q}(\mathrm{B}).$$

Wiener measure $\mathbb{Q}$ is Gaussian. $B_{[0,T]} \mapsto \frac{e^{-|t-s|}}{\|\mathrm{B}_t - \mathrm{B}_s\|}$ is SQC for any (s,t).

In fact, the Coulomb interaction is a mixture of centered Gaussians:

$$\frac{1}{x} = \sqrt{2/\pi} \int_0^\infty e^{-u^2x^2/2} \mathrm{d}u.$$

Hence $\widehat{\mathbb{Q}}_{\alpha,T}$ is an (infinite dimensional) LSGM and obeys GCI.

Now the mixture comes **inside** the exponent. The resulting Gaussian mixture representation of $\widehat{\mathbb{Q}}_{\alpha,T}$ is indexed by a deformed Poisson process on weighted time-intervals $([s,t];u)$.

Theorem (Mukherjee-Varadhan 20)

The Polaron path measure $\widehat{\mathbb{Q}}_{\alpha,T}$ has a mixture-of-Gaussian representation

$$\widehat{\mathbb{Q}}_{\alpha,T}(B_{[0,T]}) = \int \mathrm{Q}_{\xi}(B_{[0,T]})\, \widehat{\Theta}_{\alpha,T}(\mathrm{d}\xi).$$

Here $\xi = \{([s_i, t_i], u_i)\}_{i=1}^{n}$ is a point process of weighted intervals, and

$$\mathrm{dQ}_{\xi}(B_{[0,T]}) \propto e^{-\sum_{i=1}^{n} u_i^2 \|B(t_i)-B(s_i)\|^2} \mathrm{d}\mathbb{Q}(B_{[0,T]}).$$

The Fröhlich Polaron

Theorem (Mukherjee-Varadhan 20)

The Polaron path measure $\widehat{\mathbb{Q}}_{\alpha,T}$ *has a mixture-of-Gaussian representation*

$$\widehat{\mathbb{Q}}_{\alpha,T}(B_{[0,T]}) = \int \mathrm{Q}_{\xi}(B_{[0,T]})\, \widehat{\Theta}_{\alpha,T}(\mathrm{d}\xi).$$

Here $\xi = \{([s_i, t_i], u_i)\}_{i=1}^{n}$ *is a point process of weighted intervals, and*

$$\mathrm{dQ}_{\xi}(B_{[0,T]}) \propto e^{-\sum_{i=1}^{n} u_i^2 \|B(t_i)-B(s_i)\|^2} \mathrm{d}\mathbb{Q}(B_{[0,T]}).$$

- Ergodic limit $(\widehat{\Theta}_{\alpha,T}, \widehat{\mathbb{Q}}_{\alpha,T}) \to (\widehat{\Theta}_{\alpha,\infty}, \widehat{\mathbb{Q}}_{\alpha,\infty})$ as $T \to \infty$.
- Functional CLT for $\widehat{\mathbb{Q}}_{\alpha,\infty}$ [Mukherjee-Varadhan 20, Betz-Polzer 21].
  - Rigorizes path integral connection [Spohn 87, Dybalski-Spohn 20].
- Applied to show $m_{eff}(\alpha) \gtrsim \alpha^{2/5}$ [Betz-Polzer 22].
- [**S** 22,Brooks-Seiringer 22]: $\frac{\alpha^4}{C\log(\alpha)^6} \le m_{eff}(\alpha) \le C_* \alpha^4 + \alpha^{4-\varepsilon}$.

The Fröhlich Polaron

$$d\widehat{\mathbb{Q}}_{\alpha,T}(\mathrm{B}) \equiv \frac{1}{Z_{\alpha,T}} \exp\left(\alpha \int_0^T \int_0^T \frac{e^{-|t-s|}}{\|\mathrm{B}_t - \mathrm{B}_s\|}\, dt\, ds\right) d\mathbb{Q}(\mathrm{B}).$$

Theorem (Bazaes-Mukherjee-S-Varadhan 24; predicted in Landau-Pekar 1948)

If $T^{0.1} \geq \alpha \gg 1$, *then* $\mathbb{E}^{\widehat{\mathbb{Q}}_{\alpha,T}} \|\mathrm{B}_T\|^2 \leq O(T\alpha^{-4})$. *I.e.* $m_{eff}(\alpha) \geq \Omega(\alpha^4)$.

$$\mathrm{d}\widehat{\mathbb{Q}}_{\alpha,T}(\mathrm{B}) \equiv \frac{1}{Z_{\alpha,T}} \exp\left(\alpha \int_0^T \int_0^T \frac{e^{-|t-s|}}{\|\mathrm{B}_t - \mathrm{B}_s\|}\, \mathrm{d}t\, \mathrm{d}s\right) \mathrm{d}\mathbb{Q}(\mathrm{B}).$$

Theorem (Bazaes-Mukherjee-S-Varadhan 24; predicted in Landau-Pekar 1948)

If $T^{0.1} \geq \alpha \gg 1$, *then* $\mathbb{E}^{\widehat{\mathbb{Q}}_{\alpha,T}} \|\mathrm{B}_T\|^2 \leq O(T\alpha^{-4})$. *I.e.* $m_{eff}(\alpha) \geq \Omega(\alpha^4)$.

Here FKG-GCI is "technical glue" for monotonicity and local-to-global arguments. Some direct consequences:

- $\alpha \mapsto m_{eff}(\alpha)$ is strictly increasing.
- $\mathbb{E}^{\widehat{\mathbb{Q}}_{\alpha,T_1+T_2}} \|\mathrm{B}_{T_1+T_2}\|^2 \leq \mathbb{E}^{\widehat{\mathbb{Q}}_{\alpha,T_1}} \|\mathrm{B}_{T_1}\|^2 + \mathbb{E}^{\widehat{\mathbb{Q}}_{\alpha,T_2}} \|\mathrm{B}_{T_2}\|^2$.
- Universality: can replace $1/x$ by "more attractive" potentials.

Idea: find many "interval chains" $[s_1, t_1], [s_2, t_2], \ldots$ with $t_k \approx s_{k+1}$.

Conclusion

The FKG-Gaussian correlation inequality is a general tool to prove confinement of high-dimensional **unimodal** probability distributions.

Conclusion

The FKG-Gaussian correlation inequality is a general tool to prove confinement of high-dimensional **unimodal** probability distributions.

- Yields domination by certain mixtures of centered Gaussians.
- Well suited for products of symmetric unimodal densities.

Conclusion

The FKG-Gaussian correlation inequality is a general tool to prove confinement of high-dimensional **unimodal** probability distributions.

- Yields domination by certain mixtures of centered Gaussians.
- Well suited for products of symmetric unimodal densities.
- Sharp bounds for Ginzburg–Landau and Polaron models:

$$d\mu_{G,\Lambda,U}(\varphi) \equiv \frac{1}{Z_{G,\Lambda,U}} \exp\left(- \sum_{e \in E(G)} U\big(|\nabla \varphi(e)|\big) \right) \prod_{v \in \Lambda} d\varphi(v),$$

$$d\widehat{\mathbb{Q}}_{\alpha,T}(\mathrm{B}) \equiv \frac{1}{Z_{\alpha,T}} \exp\left(\alpha \int_0^T \int_0^T \frac{e^{-|t-s|}}{\|\mathrm{B}_t - \mathrm{B}_s\|}\, dt\, ds \right) d\mathbb{Q}(\mathrm{B}).$$

Conclusion

The FKG-Gaussian correlation inequality is a general tool to prove confinement of high-dimensional **unimodal** probability distributions.

- Yields domination by certain mixtures of centered Gaussians.
- Well suited for products of symmetric unimodal densities.
- Sharp bounds for Ginzburg–Landau and Polaron models:

$$d\mu_{G,\Lambda,U}(\varphi) \equiv \frac{1}{Z_{G,\Lambda,U}} \exp\left(- \sum_{e \in E(G)} U\big(|\nabla\varphi(e)|\big)\right) \prod_{v \in \Lambda} d\varphi(v),$$

$$d\widehat{\mathbb{Q}}_{\alpha,T}(\mathrm{B}) \equiv \frac{1}{Z_{\alpha,T}} \exp\left(\alpha \int_0^T \int_0^T \frac{e^{-|t-s|}}{\|\mathrm{B}_t - \mathrm{B}_s\|} \, dt \, ds \right) d\mathbb{Q}(\mathrm{B}).$$

- Anything similar without origin-symmetry? E.g. non-zero tilts.