SOME FAVORITE OLYMPIAD-STYLE PROBLEMS

MARK SELLKE

Warm-Up Problems

(1) A group of 11 scientists want to put many locks on a box and give each scientist the key to some of the locks. The goal is for no 5 scientists to be able to open every lock and hence the box, but for any 6 to be able to do so. How many locks are needed?

(2) (AoPS Intro to Counting and Probability) In how many orders can a spider put on all his socks and shoes?

(3) Show that $\gcd(\binom{n}{a}, \binom{n}{b}) > 1$ whenever $0 < a, b < n$.

(4) (Calvin Deng) A group of n frogs are hopping down $\mathbb{Z}^+$. They start at 0, and each repeatedly rolls a fair standard die to decide the length of his next jump; all jumps are to the right. Every time a frog reaches a spot, he writes his name to record that he landed there. Let k be the first spot to have all n names. Compute the expected value of k.

Hard Problems

(5) Find the chance that n uniformly random points on the sphere S^k are on a common hemisphere.

(6) Given any number of robots in various different connected $n \times n$ mazes at different starting points, a move is an attempt to move in one of the 4 directions, which fails it you hit a wall. Show that they can be moved to all be in the lower right corner.

(7) (HMIC 2016) Choose positive integers $a_1, a_2, \dots, a_n$. Let $S = \{a_1^{e_1} + a_2^{e_2} + \dots + a_n^{e_n} | e_1, e_2, \dots, e_n \in \mathbb{N}\}$. Show that there is a positive integer N only depending on n such that S contains no arithmetic progressions of length N.

(8) Given $2m$ points in the plane, prove that you can place one closed $\frac{\pi}{m}$-angle sector with center at each point to cover the whole plane.

(9) (USA TSTST 2012) There are 2010 students and 100 classrooms in the Olympiad High School. At the beginning, each of the students is in one of the classrooms. Each minute, as long as not everyone is in the same classroom, somebody walks from one classroom into a different classroom with at least as many students in it (prior to his move). This process will terminate in M minutes. Determine the maximum value of M.

(10) (Fedja, based on HMIC 2014) Prove that the maximum area you can cover with n disjoint triangles in a semicircle is achieved by the top half of a regular polygon.

(11) (MOP 2013) A tree T has some chips at each vertex. Each round, all vertices v with at least as many chips as neighbors send 1 chip to each neighbor. Prove that this is periodic with period 1 or 2.

(12) (MOP 2013) For fixed $n > 1$, let $a_1 \leq a_2 \leq \cdots \leq a_n$ be positive integers making $\sum_{i=1}^{n} \frac{1}{a_i} < 1$ with the difference as small as possible. Show a_i is defined by $a_1 = 2$ and $a_k = 1 + \prod_{i<k} a_k$. (In other words, the greedy algorithm is optimal.)

(13) (IMO 2017) A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0 are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order: The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1. A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1. The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1. Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds, she can ensure that the distance between her and the rabbit is at most 100?

(14) (Miklos Schweitzer) Let $I_1, ..., I_k \subseteq [0,1]$ be intervals. Show that $\sum_{i,j: I_i \cap I_j \neq \emptyset} \frac{1}{|I_i \cup I_j|} \geq k^2$.

(15) (Dmitry Krachun) Consider a polygon S_1 in the plane. Let S_{n+1} be the set of points x such that $|B_1(x) \cap S_n| \geq \frac{\pi}{2}$.

1. Show that S_n vanishes eventually.
2. Find a set S_1 so that some S_n has $|S_n| > 100|S_1|$.

(16) Let a, b be positive integers and suppose that $\frac{b^k-1}{a^k-1}$ is an integer for all sufficiently large integers k. Prove that b is a power of a.

(17) (USA TSTST 2018) Show that there is an absolute constant $c < 1$ with the following property: whenever $\mathcal{P}$ is a polygon with area 1 in the plane, one can translate it by a distance of $\frac{1}{100}$ in some direction to obtain a polygon $\mathcal{Q}$, for which the intersection of the interiors of $\mathcal{P}$ and $\mathcal{Q}$ has total area at most c.

(18) (RMM 2012) Each positive integer is coloured red or blue. A function f from the set of positive integers to itself has the following two properties:

(a) if $x \leq y$, then $f(x) \leq f(y)$
(b) if x, y and z are (not necessarily distinct) positive integers of the same colour and $x + y = z$, then $f(x) + f(y) = f(z)$.

Prove that there exists a positive number a such that $f(x) \leq ax$ for all positive integers x.

(19) (MOP 2013) Determine whether there is an infinite set $S \subseteq \mathbb{Z}^+$ such that $abc + 1$ is a square for all distinct $a, b, c \in S$.

(20) (Dmitry Krachun) Start with a_0 a positive integer and form a_{n+1} by taking a_n and adding the product of all the non-zero digits. Prove that some difference $a_{n+1} - a_n$ occurs infinitely often.

(21) (MOP 2013) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(m) - f(n) | m^2 - n^2$ for all $m \neq n$.

(22) (IMO 2013) Let $n \geq 3$ be an integer, and consider a circle with $n + 1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, ..., n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called beautiful if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labelled a and d does not intersect the chord joining the points labelled b and c.

Let M be the number of beautiful labelings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that $M = N + 1$.