POST-OLYMPIAD PROBLEMS

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(1) Evaluate $\int_0^{2\pi} cos^{2k}(x)dx$ using combinatorics.

(2) (Putnam) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and that for all $q \in \mathbb{Q}$ we have $f'(q) = f(q)$. Need $f(x) = ce^x$ for some c?

(3) Show that the random sum $\pm 1 \pm \frac{1}{2} \pm \frac{1}{3}...$ defines a conditionally convergent series with probability 1.

(4) (Jacob Tsimerman) Does there exist a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ with a unique critical point at 0, such that 0 is a local minimum but not a global minimum?

(5) Prove that if $f \in C^\infty([0,1])$ and for each $x \in [0,1]$ there is $n = n(x)$ with $f^{(n)} = 0$, then f is a polynomial.

(6) (Stanford Math Qual 2018) Prove that the operator $\chi_{[a_0,b_0]}\mathcal{F}\chi_{[a_1,b_1]}$ from $L^2 \to L^2$ is always compact. Here if S is a set, χ_S is multiplication by the indicator function 1_S. And $\mathcal{F}$ is the Fourier transform.

(7) Let $n \geq 4$ be a positive integer. Show there are no non-trivial solutions in entire functions to $f(z)^n + g(z)^n = h(z)^n$.

(8) Let $x \in S^n$ be a random point on the unit n-sphere for large n. Give first-order asymptotics for the largest coordinate of x.

(9) Show that almost sure convergence is not equivalent to convergence in any topology on random variables.

(10) (Noam Elkies on MO) Show that $\sum_{n=0}^{\infty} \frac{x^n}{(n!)^\alpha}$ is positive for any $\alpha \in (0,1)$ and any real x.

(11) Prove that the Galois group of the truncated exponential $T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ is A_n when $4|n$ and S_n otherwise.

(12) Find the asymptotics of $a_n = \sum_{k=0}^{n} \frac{n^k}{k!}$.

(13) (Miklos Schweitzer) Let K be a convex body in $\mathbb{R}^n$. Show that any two distinct translates of K have different centroids under a standard Gaussian distribution; in fact show this gives a homeomorphism.

(14) (Miklos Schweitzer) For $[0,1] \subset E \subset [0,+\infty)$ where E is composed of a finite number of closed intervals, we start a two dimensional Brownian motion from the point $x < 0$ terminating when we first hit E. Let $p(x)$ be the probability of the finishing point being in $[0,1]$. Prove that $p(x)$ is strictly increasing on $[-1,0)$.

(15) (Bjorn Poonen) Show that $\mathbb{R}^2$ cannot be packed with uncountably many topological 8 shapes. What about T shapes?

(16) Given an infinite subset S of $\mathbb{R}$, can you find $s \in S$ splitting it into equal cardinality left and right halves?

(17) Alice and Bob each have a countably infinite collection of boxes labelled by $\mathbb{Z}^+$. For some unknown sequence (a_k) of real numbers, Alice and Bob's kth box each contains the number a_k. Each of them will open some boxes, then predict the value of another box. Prove that they can collaborate so that at least one of them is correct.