# Random Walks

Mark Sellke

## Random Walks on $\mathbb{Z}$

**Definition 1.** A *simple random walk* on $\mathbb{Z}$ is formed by starting at 0 and adding $\pm 1$ with each term uniformly random and independent of the others.

**Problem 1.1.** *What is the probability that a simple random walk reaches* 100 *before* $-20$*?*

**Problem 1.2.** *Find the expected time for a simple random walk to reach one of* $\pm 100$.

**Problem 1.3.** *Find the expected time for a simple random walk to reach* 100.

**Problem 1.4.** *Do a simple random walk on the integers modulo* $n$*, starting from* 0*. Find the distribution of the last residue class reached by the walk.*

**Problem 1.5.** *Let* $(X_n)$ *be a* biased *random walk starting at* $X_0 = 0$*, with the differences* $X_{n+1} - X_n = 1$ *with probability* $p < \frac{1}{2}$ *and* $X_{n+1} - X_n = -1$ *with probability* $1 - p$*. Show that the chance to return to the origin is* $2p$.

**Problem 1.6.** *Why doesn't the following argument work for Problem 1.1? Let* $m$ *be the last time when* $X_m = 0$*. Then* $X_{m+1} \in \{\pm 1\}$ *with* 50-50 *probability and this determines which side we exit from, so the probability is* $\frac{1}{2}$.

**Problem 1.7.** *You are about to do a simple random walk on* $\mathbb{Z}$ *starting from* 1*. Before you start, God predicts that you will reach* 100 *before hitting* 0*. Find the new conditional probability to increase from* $n$ *to* $n+1$ *at each time, as a function of* $n$*, when* 100 *has not been hit yet.*

**Remark.** In the above, the value of 100 doesn't actually affect the answer. This means we can replace 100 with infinity to obtain a random walk *conditioned to escape to infinity.*

## Wilder Problems

**Problem 1.8.** *Consider a bounded* discrete harmonic *function* $f : \mathbb{Z}^d \to [0, 1]$*, meaning that every vertex has the average value of its neighbors. Show that* $f$ *is constant.*

**Problem 1.9.** *(December 2017 TST) Find all functions* $f : \mathbb{Z}^2 \to [0, 1]$ *such that for any integers* $x$ *and* $y$*,*

$$f(x, y) = \frac{f(x-1, y) + f(x, y-1)}{2}.$$

*However show that this is false if* $[0, 1]$ *is replaced with* $\mathbb{R}^+$.

**Problem 1.10** (Calvin Deng). *We roll two standard $n$-sided dice infinitely many times each, and compute the sequences $S_1, S_2$ of partial sums for each die. Let $s$ be the smallest positive integer to appear in both sequences. Compute the expected value of $s$.*

**Problem 1.11.** *On average, how many coin flips will it take for the string* 101010101011000101 *to appear? Perhaps surprisingly you can do this with minimal arithmetic.*

**Problem 1.12.** *You probably know that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, and that the alternating harmonic series $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$ converges. What if the signs $\pm 1$ are generated by independent, unbiased coin flips?*

**Problem 1.13** (Wilson's Algorithm). *Let $G$ be a connected graph. A simple random walk on $G$ from some starting vertex is defined by moving to a uniformly random neighbor at each time step. A* loop erased random walk *(LERW) on $G$ is defined by performing a simple random walk and drawing the path of the walk, but erasing any loops you make when you hit the path that you have currently drawn. So at any time, you have a simple path from the starting position to the current position. (Ask me for more details if that's confusing.)*

*Now perform the following algorithm to generate a spanning tree $T$ of $G$.*

1. *Pick an arbitrary root vertex $v$ for $T$.*
2. *Pick an arbitrary vertex $w$ and perform a LERW from $w$ until hitting $v$. Add this path to $T$.*
3. *Repeat the previous step, starting a LERW from a vertex not already included in $T$ and stopping when it first hits $T$, then adding the currently drawn path in as a new branch of $T$. Repeat until all vertices are used, so $T$ is a spanning tree.*

*Prove that the tree $T$ generated is uniformly random from the set of spanning trees of $G$, regardless of the arbitrary choices. Visit* `https://bl.ocks.org/mbostock/11357811` *to see this in action!*