STAT 212 Problem Set 3.

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Due: Friday, March 7th at 11:59PM

Instructions: Collaboration with your classmates is encouraged. Please identify everyone you worked with at the beginning of your solution PDF (e.g. Collaborators: Alice, Bob). Your solutions should be *written* entirely by you, even if you collaborated to *solve* the problems. The first person to report each typo in this problem set (by emailing me and Somak) will receive 1 extra point; more serious mistakes will earn more points.

In all problems below, B_t denotes standard Brownian motion.

1. In this problem, you will complete the construction of Brownian motion from class.

   (i) Show by induction on k that for each $k \geq 1$, the random function $B_t^{(k)}$ obeys the distributional conditions of Brownian motion, restricted to the set of times $2^{-k}\mathbb{Z} \cap [0,1]$.

   (ii) Show that the limiting function $B_t = \lim_{k\to\infty} B_t^{(k)}$ we constructed obeys the distributional conditions of Brownian motion at *all* real times $t \in [0,1]$. (Hint: in approximating real numbers by dyadic rationals, it may help to recall that almost sure convergence of scalar random variables implies convergence in distribution.)

2. In this problem, you will investigate the maximum absolute value of Brownian motion.

   (i) Prove that $f \mapsto \max_{t\in[0,1]} |f(t)|$ is measurable on $f \in C([0,1])$, relative to the usual σ-field on the latter space. Conclude that $\max_{t\in[0,1]} |B_t|$ is a bona-fide random variable.

   (ii) Prove that $\mathbb{E}[\max_{t\in[0,1]} |B_t|] < \infty$.

   (iii) Establish the stronger estimate

   $$\mathbb{P}\Big[\max_{0\leq t\leq 1} |B_t| > \lambda\Big] \leq Ce^{-c\lambda^2}$$

   for universal constants $C, c > 0$.

   (iv) Show that $|B_t|$ almost surely attains its maximum value on $t \in [0,1]$ at **exactly** one value of t.

   Hints: For part (ii), the estimates involved in constructing Brownian Motion in class may be useful. For part (iii), the reflection principle may help. For part (iv), you may want to show that $\mathbb{P}[\max_{0\leq t\leq a} |B_t| = \max_{a\leq t\leq 1} |B_t|] = 0$ for each fixed $a \in [0,1]$.

3. Planar Brownian motion is defined to be a random continuous function $\vec{B}_t = (B_t^1, B_t^2)$ in which B^1, B^2 are IID standard Brownian motions.

(i) Show that for any fixed $\theta \in [0, 2\pi)$, the rotated process

$$\vec{B}_t^{(\theta)} = \big(\cos(\theta)B_t^1 + \sin(\theta)B_t^2, -\sin(\theta)B_t^1 + \cos(\theta)B_t^2\big)$$

also has the law of planar Brownian motion. Thus, planar Brownian motion is rotationally invariant.

(ii) Show that $\vec{B}_t$ is a continuous $\mathbb{R}^2$-valued martingale relative to its natural filtration $\mathcal{F}_t = \sigma(\{(B_s^1, B_s^2) : 0 \le s \le t\})$

(iii) Show that $\tau = \inf\{t \ge 0 : |\vec{B}_t| = 1\}$ is a stopping time (relative to the natural filtration).

4. **(Extra credit)** In this problem, you will investigate the *modulus of continuity* of Brownian motion. In general, given a continuous function $f : [0,1] \to \mathbb{R}$, the modulus of continuity is the function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\omega(\varepsilon) = \sup_{\substack{x,y\in[0,1] \\ |x-y|\le\varepsilon}} |f(x) - f(y)|.$$

For any continuous f, one has $\lim_{\varepsilon\to 0} \omega(\varepsilon) = 0$. Below, we let ω be the modulus of continuity of standard Brownian motion (thus ω is random).

(i) Show there is a universal constant $c > 0$ such that $\mathbb{P}[\omega(\varepsilon) \ge c\sqrt{\varepsilon \log(1/\varepsilon)}] \ge 1/2$ for all sufficiently small ε.

(ii) Show there is a universal constant $C > 0$ such that $\mathbb{P}[\omega(\varepsilon) \le C\sqrt{\varepsilon \log(1/\varepsilon)}] \ge 1/2$ for all sufficiently small ε.

(iii) Show there are random constants C, c (depending on the Brownian motion) such that

$$\omega(\varepsilon) \in [c\sqrt{\varepsilon \log(1/\varepsilon)}, C\sqrt{\varepsilon \log(1/\varepsilon)}]$$

holds simultaneously for all $\varepsilon \in (0, 1/2)$. (In fact, it is a famous result of Lévy that $\omega(\varepsilon) \sim \sqrt{2\varepsilon \log(1/\varepsilon))}$ as $\varepsilon \downarrow 0$, almost surely.)

(Hint: extending the expected value estimate from class, it may help to argue that the maximum of n IID standard Gaussians is typically of order $\Theta(\sqrt{\log n})$.)

5. **(Extra credit)** This problem investigates the Fourier expansion of Brownian motion.

(i) Let $Y_t = B_t - tB_1$ for $t \in [0,1]$, so that $Y_0 = Y_1 = 0$ almost surely. Show that Y_t is a centered Gaussian process with covariance $\mathbb{E}[Y_sY_t] = s(1-t)$ for $0 \le s \le t \le 1$. (Y_t is called a *Brownian bridge*.)

(ii) For each $n \ge 0$, show that

$$a_n = \int_0^1 \sin(\pi n t) Y_t dt$$

is a centered Gaussian, and determine the covariance function $\mathbb{E}[a_n a_m]$. (Hint: for the first assertion, it may help to consider Riemann sum approximations.)

(iii) Show that $(a_n)_{n\geq 1}$ are jointly independent.

(iv) Give two formulas for $\mathbb{E}[\int_0^1 Y_t^2 dt]$, one using Fubini directly and the other using Fourier series (i.e. Plancherel's identity). Deduce that $\sum_{k\geq 1} 1/k^2 = \pi^2/6$.