STAT 212 Problem Set 6.

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Due: Wednesday, April 30th at 11:59PM

Instructions: Collaboration with your classmates is encouraged. Please identify everyone you worked with at the beginning of your solution PDF (e.g. Collaborators: Alice, Bob). Your solutions should be *written* entirely by you, even if you collaborated to *solve* the problems. The first person to report each typo in this problem set (by emailing me and Somak) will receive 1 extra point; more serious mistakes will earn more points.

In all problems below, B_t denotes standard Brownian motion, and $\mathcal{F}_t$ the associated filtration.

1. For the martingale CLT from class, we assumed the total variance $V = \sum_{k=1}^{m(n)} \mathbb{E}[X_k^2|\mathcal{F}_{n,k-1}]$ equals 1 almost surely. Show the martingale CLT from class continues to hold if V converges in probability to 1. (**Hint** given on the last page.)

2. Consider d-dimensional Brownian motion $\vec{B}_t$ for $d \geq 3$, started *not* at the origin.

   (a) Prove that
   $$X_t = |\vec{B}_t|^{2-d}$$
   is a local martingale, but not a true martingale. (**Hint** given on the last page.)

   (b) Deduce that $\vec{B}_t$ almost surely never hits the origin.

   (c) Show that in fact $\vec{B}_t$ must wander off to infinity and eventually never come back to any given bounded region (i.e. $\vec{B}_t$ is transient).

3. Here you will construct spherical Brownian motion in $\mathbb{R}^d$, for $d \geq 2$. (Another construction in the $d = 2$ case was also given in the previous homework.) For $\|x\| > 0$, let $P_x^\perp$ be the projection matrix onto $x^\perp$, given explicitly by $P_x^\perp = I_d - \frac{xx^\top}{\|x\|^2}$.

   You may use below that existence and uniqueness for SDE solutions as in class continues to hold (with exactly the same proof) in higher dimensions, for equations of the form
   $$dX_t = \sigma(X_t, t)dB_t + v(X_t, t)dt$$
   with $\sigma : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ and $v : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$.

   (a) Show that there exists a Lipschitz function $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ such that $\sigma(x) = P_x^\perp$ for all $\|x\| \geq 1$.

   (b) Consider the natural attempt at spherical Brownian motion given by initializing $X_0 = (1, 0, \ldots, 0)$ and setting
   $$dX_t = \sigma(X_t)dB_t.$$
   for a standard d-dimensional Brownian motion B_t. Show this SDE has a unique solution.

(c) Use Ito's formula to give an integral expression for $\|X_t\|^2$ and conclude that $\|X_t\| > 1$ for all $t > 0$, almost surely. (Thus, this attempt did not work.)

(d) Find a constant b (possibly depending on the dimension) such that

$$dX_t = \sigma(X_t)dB_t - bX_t dt$$

Does stay confined to the unit sphere for all time. (Caution: after finding b such that $d|X_t|^2 = 0dt + 0dB_t$ **assuming** $|X_t| = 1$, you still need to show that the actual solution X_t remains on the sphere. For this, you might find an ODE solved by $|X_t|^2$ and argue uniqueness of solutions.)

4. (**Extra Credit**) In class, we used the Lindeberg method to obtain central limit theorems for independent sums and martingales. Here you will investigate further and obtain quantitative bounds.

Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth function with $\varphi(x) = 0$ for all $|x| \geq 1$ and $\int_{-1}^{1} \varphi(x)dx = 1$. (You may assume such a "bump function" exists, and are encouraged to look them up if unfamiliar.)

Let $\varphi_\epsilon(x) = \varphi(x/\epsilon)/\epsilon$, so that

$$\int_{-\infty}^{\infty} \varphi_\epsilon(x)dx = 1$$

for all ϵ. Note that for each $j \geq 0$ there is $C_j < \infty$ such that the j-th derivative of φ_ϵ is at most

$$\sup_{x\in\mathbb{R}} |\varphi_\epsilon^{(j)}(x)| \leq C_j \epsilon^{-1-j}.$$

(a) Consider $\epsilon = n^{-a}$ for small $a \in (0,1)$ (you can choose!) and let $f_{\epsilon,u}$ be the convolution of φ_ϵ with $1_{[u,\infty)}$, i.e.

$$f_{\epsilon,u}(x) = \int_u^{\infty} \varphi_\epsilon(y)dy.$$

Consider an independent sum $S_n = X_1 + \cdots + X_n$ with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = 1$ for each i, and $\mathbb{E}[|X_i|^3] \leq C$ for all i. Using the Lindeberg method from class, show a convergence rate

$$\sup_u |\mathbb{E}[f_{\epsilon,u}(S_n/\sqrt{n}) - f_{\epsilon,u}(z)]| \leq O(n^{-b})$$

for $b > 0$, where z is a standard Gaussian.

(b) Deduce uniform convergence (i.e. convergence in L^∞) of the cumulative distribution functions, again at a rate n^{-a} for some $a > 0$. (**Hint** given on last page.)

(c) Deduce an improved convergence rate for symmetric X_i with uniformly bounded 4-th moment, using matching of the 3-rd moments.

Hints

- For 1: fix small η and apply the martingale CLT from class to a stopped martingale with slightly rescaled last increment, which has total variance exactly $1-\eta$ whenever $V \geq 1-\eta$ holds for the original martingale. Then use the L^2 maximal inequality to show this is a sufficiently good approximation.
- For 2(a): to show it is not a true martingale, write an integral expression for $\mathbb{E}[X_t]$ and show it tends to 0 as $t \to \infty$.
- For 4(c): you may want to use the boundedness of the Gaussian density here.