Statistics 212: Lecture 10 (March 3, 2025)

Strong Markov Property and Reflection Principle

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1 Recap of Stopping Times

Recall that $T \geq 0$ is a stopping time with respect to $(\mathscr{F}_t)$ if $\{T \leq t\} \in \mathscr{F}_t \forall t$.

Def: $(\mathscr{F}_t)$ is a right continuous filtration if $\cap_{s>t}\mathscr{F}_s = \mathscr{F}_t$.
Recall that any $\mathscr{F}_t$ can be made right continuous by defining $\mathscr{F}_t^+ = \cap_{s>t}\mathscr{F}_s$

2 Strong Markov Property and the Reflection Principle

2.1 Warm Up Question

Let $T_a = \inf\{t \geq 0 : B_t \geq a\}$ and $\tilde{T}_a = \{t \geq 0 : B_t > a\}$.

Which are stopping times? What if we have an arbitrary probability measure on $C([0,\infty))$ and its natural filtration?

We all agree that $T_a = \inf\{t \geq 0 : B_t \geq a\}$ is a stopping time. Why?
Consider the event $T_a \leq x \iff \sup_{s \leq t} B_t \geq a$ which (by homework) is $\mathscr{F}_t$ measurable.

We mostly agree that $\tilde{T}_a = \{t \geq 0 : B_t > a\}$ is a stopping time with a minor caveat.

First observe that $T_a = \tilde{T}_a$ almost surely because Brownian Motion is positive somewhere in the interval $[0,\epsilon]$ for all $\epsilon > 0$ almost surely. So the answer is yes if $\mathscr{F}_0$ includes all measure 0 events.

Notice that $\tilde{T}_a$ is always $\mathscr{F}_t^+$ measurable for any random continuous path (not necessarily Brownian motion). However if you just have the natural filtration on an arbitrary random path $\tilde{T}_a$ need not be a stopping time. Consider the example of a path which is a deterministic path up to time t_0 when it achieves our desired value a, and then the path takes on a random (crucially positive or negative slope). Then, without looking a little into the future, one can't know for sure whether $\{\tilde{T}_a = t_0\} \in \mathscr{F}_{t_0}$.

2.2 Strong Markov Property

Thrm: Let T be a stopping time with respect to $\mathscr{F}_t^+$ for Brownian motion. Let $W_t = B_{T+t} - B_T$ for all $t \geq 0$. Then $(W_t)_{t\geq 0}$ is Brownian motion and independent of the stopped sigma field $\mathscr{F}_T^+ = \{A \in \sigma(B_s) : A \cap \{T \leq$

$t\} \in \mathscr{F}_t^+ \forall t\}$

2.2.1 Proof

(a) Assume T is almost surely one of $t_1 < t_2 < t_3, \ldots.$ and let $W_t^i = B_{t+t_i} - B_{t_i}$. Fix $A \in \mathscr{F}_t^+$ and $E \in$ Borel $C([0,\infty))$ Consider

$$\mathbb{P}[(W_t)_{t\geq 0} \in E \text{ and A holds}] = \sum_{i=1}^{\infty} \mathbb{P}(W_t \in E \text{ and A holds and } T = t_i)$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(W_t^i \in E \text{ and } A \cap T = t_i)$$

Notice that W_t^i is independent of $\mathscr{F}_{t_i}^+$ and $A \cap T = t_i \in \mathscr{F}_{t_i}^+$ so W_t is just standard Brownian motion

$$= \mathbb{P}(W_t \in E) \sum_{i=1}^{\infty} \mathbb{P}(A \cap T = t_i)$$

$$= \mathbb{P}(W_t \in E) P(A)$$

The intuition is that if I stop my Brownian motion at some deterministic time, I'll get independent Brownian motion as I look into the future so I'm just applying this to a countable collection of values my stopping time might take on

(b) Approximate a general stopping time with this countable set.
Given a stopping time T, define $T_n = \frac{\lceil 2^n T \rceil}{2^n} = \frac{m+1}{2^n}$ if $T \in (\frac{m}{2^n}, \frac{m+1}{2^n}]$. Then observe that

$$T_1 \geq T_2 \geq T_3 \ldots . T_n \downarrow T$$

Then the previous step applies to each of the T_ns since each T_n is itself a stopping time (left as a brief verification).
Then

$$W_t^{(n)} = B_{T_n + t} - B_{T_n}$$

Now we claim that

$$\lim_{n\to\infty} W_t^{(n)} = W_t$$

almost surely in $C([0,\infty))$, which follows since $T_n \to T$ and Brownian motion is continuous (comes from a standard real analysis style proof, where we're using the fact that we have a nice metric and topology on our space). Then W_t is Brownian motion since each of the $W_t^{(n)}$s are (since almost sure convergence implies convergence in distribution).
Now, moving to showing the desired independence, we have that $W_t^{(n)}$ is independent of $\mathscr{F}_{T_n} \supseteq \mathscr{F}_T^+$ since $T_n \geq T$. Then each $W_t^{(n)}$ is independent of $\mathscr{F}_T^+$ so the limit W_t is also independent of $\mathscr{F}_T$. Thus the proof is complete!

2.3 The Reflection Principle

2.3.1 Motivating Question

What is the distribution of $T_a = \inf\{t \geq 0 : B_t \geq a\}$? By inverting things, this also gives you the maximum value of Brownian motion (ie $\max\{B_t : 0 \leq t \leq t_0\}$.

2.3.2 Reflection Principle

Let T be a stopping time for Brownian Motion and consider

$$\tilde{B}_t = \begin{cases} B_t \text{ for } 0 \le t \le T \\ 2B_T - B_t \text{ for } t \ge T \end{cases}$$

The claim is that for any stopping time T, this new process $\tilde{B}_t$ is also Brownian motion.

2.3.3 Proof

Let Z_t be Brownian Motion and W_t be an independent Brownian Motion. Then consider the map

$$t \mapsto \begin{cases} Z_t \text{ for } t \le T \\ Z_T + W_{t-T} \text{ for } t \ge T \end{cases}$$

Then the claim follows from the Strong Markov Property, which tells us

$$B_t = \begin{cases} Z_t \text{ for } t \le T \\ Z_T + W_{t-T} \text{ for } t \ge T \end{cases}$$

is standard Brownian motion and

$$\tilde{B}_t = \begin{cases} Z_t \text{ for } t \le T \\ Z_T - W_{t-T} \text{ for } t \ge T \end{cases}$$

is also standard Brownian motion.

2.3.4 Return to Our Motivating Questions

Intuition: Let's suppose we achieve a at time $T_a = t$ and then reflect our Brownian motion there. Then exactly one path will be above a and one path will be below a. If we never hit a, both our reflected paths will be below a.

Consider

$$\mathbb{P}[T_a \le t] = \mathbb{P}[B_t \ge a] + \mathbb{P}[\tilde{B}_t \ge a] = 2\mathbb{P}[B_t \ge a] = 2[Z \ge \frac{a}{\sqrt{t}}]$$

The first equality follows from a statement about indicators: $\mathbb{I}_{T_a \le t} = \mathbb{I}_{B_t \ge a} + \mathbb{I}_{\tilde{B}_t \ge a}$ for $Z \sim \mathcal{N}(0,1)$ which directly gives the distribution of T_a. Now, for the max, we have

$$\{T_a \le t\} = \{\max_{0 \le s \le t} B_s \ge a\}$$

which also gives the CDF for $\max_{0 \le s \le t} B_s$. If we're looking at tail probabilities, we can do better than a union bound by using this strategy!

2.4 Continuous Time Martingales

Def: A family of random variables indexed over continuous time $(X_t)_{t \in (0,\infty)}$ adapted to a continuous time filtration $(\mathscr{F}_t)_{t \ge 0}$ is a martingale if

(a) $\mathbb{E}|X_t| < \infty$

(b) $\mathbb{E}[X_t|\mathscr{F}_s] = X_s \forall s \le t$

2.4.1 Useful Facts

Brownian motion is a martingale.

A weaker version of optional stopping using the reflection principle: $\frac{B_t+\tilde{B}_t}{2} = B_T$ for $t \geq T$ which implies the stronger statement that $\mathbb{E}[X_t|\mathscr{F}_T] = X_T$ for $t \geq T$ and T a stopping times.

2.4.2 Continuous Time Optional Stopping

Claim Let (X_t) be a continuous martingale and T a stopping time, and $T \leq c$ for some constant $c \in \mathbb{R}$ almost surely and $\mathbb{E}[\sup_{0\leq t\leq c+1}|X_t|] < \infty$ then

(a) $\mathbb{E}[X_T] = X_0$

(b) if $S \leq T$ are stopping times, then $\mathbb{E}[X_T|\mathscr{F}_S] = X_S$

Proof Idea: Let $T_n = \frac{\lceil 2^n T\rceil}{2^n}$. Then $\mathbb{E}[\sup_{0\leq t\leq c+1}|X_t|] < \infty$ and Dominated convergence imply that $X_{T_n} \to X_T$ almost surely in L^1. We know that $\mathbb{E}[X_{T_n}] = X_0$ by the discrete optional stopping theorem and so by L^1 convergence we can say that $\mathbb{E}X_T = X_0$. The second part is left as an exercise (create a new process that starts at S)

2.5 Logistics

Midterm format: 6 problems and 1 extra credit problem.