Statistics 212: Lecture 11 (March 5, 2025)

Brownian Motion as a Continuous Martingale

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1 Midterm Logistics

As announced on Canvas, the midterm will be held during class next Monday. It will have 6 problems and 1 bonus question. The focus will be understanding and applying the material from class, rather than technical details (e.g. exam will not require you to use the $\pi - \lambda$ theorem). Content will be everything up to this lecture's material on Wald's Lemma.

2 Wald's Lemma

Wald's Lemma provides the continuous time analog for optional stopping with Brownian motion, which makes sense since we know from previous lectures that Brownian motion is a continuos time martingale.

Lemma 2.1 (Wald)**.** *Let B_t be standard Brownian motion and T a stopping time with $\mathbb{E}[T] < \infty$. Then,*

(a) $\mathbb{E}[B_T] = 0$

(b) $\mathbb{E}[B_T^2] = \mathbb{E}[T]$.

Proof. *(a)* Let $M := \sup_{0\le t\le T} |B_t|$, and suppose $\mathbb{E}[M] < \infty$ (which we prove below in Lemma 2.2). Define $T_n := T \wedge n$, which is bounded for all n. Then, we can apply the optional stopping result from last lecture to conclude $\mathbb{E}[B_{T_n}] = 0$. Observe that $B_{T_n} \xrightarrow{a.s.} B_T$ as $n \to \infty$ by construction. Since $|B_{T_n}|, |B_T| \le M$, we can then apply DCT to conclude that $\mathbb{E}[B_T] = 0$, as desired. □

Lemma 2.2. *Let $M := \sup_{0\le t\le T} |B_t|$. If $\mathbb{E}[T] < \infty$, then $\mathbb{E}[M] < \infty$.*

Proof. Define $M_i := \max_{t\in[i,i+1]} |B_t - B_i|$, such that $M_1, M_2, \ldots$ are iid with finite expectation (since we know that each has a subgaussian tail bound from last lecture). Observe that $M \le \sum_{i=1}^{\lceil T\rceil} M_i$, in which case

$$\mathbb{E}[M] \le \mathbb{E}\left[\sum_{i=1}^{\lceil T\rceil} M_i\right] \tag{1}$$

$$= \mathbb{E}\left[\sum_{i=1}^{\infty} M_i \mathbb{1}\{i \le T\}\right] \tag{2}$$

$$= \mathbb{E}[M_1] \sum_{i=1}^{\infty} \mathbb{E}\left[\mathbb{1}\{i \le T\}\right], \qquad (M_i \perp \mathbb{1}\{i \le T\})$$

where we know that M_i and $\mathbb{1}\{i \le T\}$ are independent by the Strong Markov Property, as M_i only depends on B_t for $t > i$. Since $\mathbb{E}[M_1] < \infty$ and $\sum_{i=1}^{\infty} \mathbb{E}\left[\mathbb{1}\{i \le T\}\right] \approx \mathbb{E}[T] \pm \mathcal{O}(1) < \infty$, we can conclude that $\mathbb{E}[M] < \infty$. □

Proving part (b) of Wald's lemma is a little more tricky. The basic idea is that $B_t^2 - t$ is a martingale (Turns out, this is an important fundamental property of Brownian motion), since for $s < t$, we have that

$$\begin{aligned} B_t^2 &= (B_s + (B_t - B_s))^2 = B_s^2 + 2B_s(B_t - B_s) + (B_t - B_s)^2 &(3)\\ \implies \quad \mathbb{E}[B_t^2|\mathscr{F}_s] &= \mathbb{E}[B_s^2|\mathscr{F}_s] + 2\mathbb{E}[B_s(B_t - B_s)|\mathscr{F}_s] + \mathbb{E}[(B_t - B_s)^2|\mathscr{F}_s] &(4)\\ &= B_s^2 + 2B_s\mathbb{E}[B_t - B_s|\mathscr{F}_s] + \text{Var}(B_{t-s}) &(5)\\ &= B_s^2 + t - s. &(6)\end{aligned}$$

First, we show that the desired result holds when we have a bounded stopping time $T_n \equiv T \wedge n$.

Lemma 2.3. $\mathbb{E}[B_{T_n}^2] = \mathbb{E}[T_n]$

Proof. By definition, we see that $T_n + 1 \leq n + 1$ almost surely. Then,

$$\mathbb{E}\left[\sup_{t \leq T_n+1} B_t^2\right] \leq \mathbb{E}\left[\sup_{t \leq n+1} B_t^2\right] \leq \infty, \tag{7}$$

where the last inequality follows from subgaussian tail bound we have from the reflection principle. Then, the conditions for continuous time optional stopping (from last lecture) are satisfied by $(B_t^2 - t)_{t \geq 0}$ and T_n, and we have that $\mathbb{E}[B_{T_n}^2 - T_n] = B_0^2 - 0 = 0 \implies \mathbb{E}[B_{T_n}^2] = \mathbb{E}[T_n]$, as desired. □

Now, all that remains is to extend this result to general stopping times T with finite expectation. The main lift will be to show that $\mathbb{E}[M^2] < \infty$, in which case the dominated convergence argument as in the proof of part (a) can be applied.

Lemma 2.4. *Let* $M_n := \sup_{t \leq T \wedge n} |B_t|$. *Then,* $\mathbb{E}[M_n^2] \leq 4\mathbb{E}[B_{T \wedge n}^2]$.

Proof. Recall the L^p maximal inequality, specialized to the case when $p = 2$ (Lemma 1.4, Lecture 4 Scribe Notes): *If* $X_1, ..., X_k$ *is a* L^2 *bounded (discrete) martingale, then* $||\sup_{j \leq k} |X_j|||_2 \leq 2\sup_j ||X_j||_2 = 2||X_k||_2$.

The statement extends easily to continuous time, were we apply the usual discretization argument using 2^{-m}-length time increments by defining $M_n^{(m)} := \sup_{t \leq T \wedge n,\, t \in 2^{-m}\mathbb{Z}} |B_t|$ such that $M_n^{(m)} \uparrow M_n$ as $m \to \infty$. Then, applying the discrete version of L^2 maximal inequality to each $M_n^{(m)}$, we have

$$\underbrace{||M_n^{(m)}||_2^2}_{=\mathbb{E}[(M_n^{(m)})^2]} \leq 4 \underbrace{||B_{T \wedge n}||_2^2}_{=\mathbb{E}[B_{T \wedge n}^2]} \tag{8}$$

for all m, in which case monotone convergence implies that $\mathbb{E}[M_n^2] \leq 4\mathbb{E}[B_{T \wedge n}^2]$, as desired. □

Lemma 2.5. *Let* $M := \sup_{0 \leq t \leq T} |B_t|$. *If* $\mathbb{E}[T] < \infty$, *then* $\mathbb{E}[M^2] < \infty$.

Proof. We have the following chain of inequalities:

$$\mathbb{E}[M_n^2] \underbrace{\leq}_{\text{Lemma 2.4}} 4\mathbb{E}[B_{T \wedge n}^2] \underbrace{=}_{\text{Lemma 2.3}} 4\mathbb{E}[T \wedge n] \leq 4\mathbb{E}[T] < \infty. \tag{9}$$

Since $M_n \uparrow M$, taking the limit $n \to \infty$ and applying dominated convergence implies $\mathbb{E}[M^2] < \infty$. □

The remainder of the proof of statement (b) of Wald's Lemma then follows by dominated convergence on the bounded stopping time version of the statement, where

$$\begin{aligned} \lim_{n \to \infty} \mathbb{E}[B_{T_n}^2] &= \lim_{n \to \infty} \mathbb{E}[T_n] &&\text{(Lemma 2.3)}\\ &= \mathbb{E}[T] &&(\text{DCT}, \quad T_n \leq T)\\ \lim_{n \to \infty} \mathbb{E}[B_{T_n}^2] &= \mathbb{E}[B_T^2], &&(\text{DCT}, \quad |B_{T_n}^2|, |B_T^2| \leq M)\end{aligned}$$

allowing us to conclude that $\mathbb{E}[T] = \mathbb{E}[B_T^2]$. This completes our proof of Wald's Lemma ■

Finally, we consider some applications of Wald's Lemma.

Proposition 2.6. *Let* $T := \inf_{t\geq 0}\{B_t = 1\}$. *Then,* $\mathbb{E}[T] = \infty$.

Proof. Suppose, for a contradiction, that $\mathbb{E}[T] < \infty$. Wald's Lemma then implies $\mathbb{E}[B_T] = 0$. This must be a contradiction, since $B_T = 1$ almost surely. ☐

However, it turns out adding one more "boundary" to define the stopping time results in T's tails decaying exponentially.

Proposition 2.7. *Let* $a, b > 0$ *and define* $T := \inf\{t \geq 0 : B_t = -a \text{ or } B_t = b\}$. *Then,* $\mathbb{E}[T] < \infty$, *and for any* $k \in \mathbb{Z}_+$ $\mathbb{P}\{T \geq k\} \leq e^{-ck}$ *for some* $c := c(a, b) > 0$,

Proof. Let $W_t^i := B_{i+t} - B_i$ for $i \in \mathbb{Z}_+$ and $t \in [0, 1]$. Observe that the following statement is true:

$$\text{If } \left\{\sup_{0\leq t\leq 1} W_t^i \geq a + b\right\} \text{ or } \left\{\inf_{0\leq t\leq 1} W_t^i \leq -a - b\right\}, \text{ then } T \leq i + 1, \tag{10}$$

as either statement implies that our process' value at time i and $i + t$ differs by at least $a + b$. If we define S as the first i that obeys at least one of the above two conditions, it then follows that $S + 1 \geq T$ almost surely. Noting that S is a geometric random variable, we conclude $\mathbb{E}[T] \leq \mathbb{E}[S] + 1 < \infty$. The tail bound also follows from the fact that $\mathbb{P}\{T \geq k\} \leq \mathbb{P}\{S \geq k - 1\}$ and using the CDF of a geometric random variable. ☐

Note that $\mathbb{P}\{B_T = a\} = \frac{b}{a+b}$ in the above example via a symmetry/reflection argument, as $B_0 = 0$. Using the fact that $(B_t + a)(B_t - b) - t$ is a martingale, one can also deduce from Wald's Lemma that $\mathbb{E}[T] = \mathbb{E}[B_T^2] = ab$.

3 Preview of Donsker's Theorem

The past few lectures, we have focused on coming up with continuous-time analogs to the tools we have for discrete time martingales. In a similar spirit, Donsker's Theorem can be interpreted as the continuous time/random path analog to the central limit theorem.

Theorem 3.1 (Donsker)**.** *Let* $A_1, A_2, \ldots$ *be iid, where* $\mathbb{E}[A_1] = 0$ *and* $Var(A_1) = 1$. *Consider a random path* X_t^n *for* $n \geq 1$ *large, where* $X^n_{\frac{k}{n}} = \frac{A_1 + \ldots + A_k}{\sqrt{n}}$, *and the remainder of* X_t^n *for* $t \notin \frac{1}{n}\mathbb{Z}_+$ *is defined by interpolating. Then,*

$$(X_t^n)_{t\geq 0} \xrightarrow{d} (B_t)_{t\geq 0} \quad \text{as } n \to \infty. \tag{11}$$

Proof Idea. Construct a stopping time T (for Brownian motion) with finite expectation, such that $B_T \overset{d}{=} A_1$. By Wald's Lemma, $\mathbb{E}[T] = \mathbb{E}[B_T^2] = \mathbb{E}[A_1^2] = \text{Var}(A_1) = 1$. The idea will be to note that $A_1 + \ldots A_k = B_{T_1 + \ldots + T_k}$, in which case applying LLN on the sums of T_i will imply that $T_1 + \ldots + T_k$ behaves almost deterministically.