Statistics 212: Lecture 14 (March 26, 2025)

Progressively Measurable Processes, Ito Isometry

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1 Logistics

Next class (3/31): guest lecture on random matrices (Kevin Yang)

2 Stochastic Differential Equations and Stochastic Integration

Consider the stochastic differential equation (SDE)

$$dY_t = \sigma_t \mathrm{d}B_t + v_t \mathrm{d}t,$$

where σ_t and v_t are adapted processes with respect to filtration $\mathscr{F}_t$, such as that generated by BM.

2.1 SDE Examples

- If $\sigma_t = f(Y_t)$ and $v_t = g(Y_t)$, then Y_t is Markovian.
- Picking $\sigma_t = \begin{cases} 1, & \text{if } \max_{s\le t} |B_s| < 1 \\ 0, & \text{if } \max_{s\le t} |B_s| \ge 1 \end{cases}$, and $v_t = 0$, the SDE represents a BM stopped at ± 1.

2.2 Simulating SDEs

To simulate SDEs, we can discretize time into steps of size ϵ. The resulting integral approximations are as follows:

$$\int_0^t v_s \mathrm{d}s = \epsilon(v_0 + v_\epsilon + \ldots + v_{t-\epsilon})$$

$$\int_0^t \sigma_s \mathrm{d}B_s = \sigma_0 \underbrace{\sqrt{\epsilon} Z_0}_{B_\epsilon - B_0} + \sigma_\epsilon \underbrace{\sqrt{\epsilon} Z_1}_{B_{2\epsilon} - B_\epsilon} + \ldots,$$

where Z_i are iid Gaussian.

Central Question: How can we justify an $\epsilon \to 0$ limit in the above? Riemann integral is not defined since Brownian motion has unbounded variation.

Let's see some technical difficulties in the theory, which are already present with the special case of the Paley-Wiener integral.

2.3 Technical Difficulties with the Paley-Wiener Integral

Let $\sigma_t = F(t)$, a deterministic $L^2([0,1])$ function. We defined $\int_0^1 F(t)\mathrm{d}B_t$ last time via an L^2 isometry.

Remark: $\forall F \in L^2([0,1])$ fixed, $\int_0^1 F(t)\mathrm{d}B_t$ is a.s. finite, and is a centered Gaussian with variance $\|F\|_{L^2}^2$. However, $\int_0^1 F(t)\mathrm{d}B_t$ may not be finite $\forall F$ simultaneously; if $F \in L^2([0,1])$ depends on $B_{[0,1]}$, the integral may diverge.

Example. We demonstrate the latter statement in the above remark with the following example. Construct functions

$$f_j(t) = \sqrt{2}\sin(\pi j t).$$

Then $\{f_1, f_2, \dots\}$ is an orthonormal basis for $L^2([0,1])$.
By the L^2 isometry from before,

$$g_j = \int_0^1 f_j(t)\mathrm{d}B_t$$

are iid $\mathcal{N}(0,1)$.
Let n_j be smallest n such that $g_n \geq j$ and $n_j > n_{j-1}$. Each n_j is finite since IID Gaussian sequences are almost surely unbounded.
Using indices n_j, construct $F = \sum_{j=1}^{\infty} \frac{f_{n_j}}{j}$. Then we have that $||F||_2^2 = \sum \frac{1}{j^2} < \infty$, yet the integral $\int_0^1 F(t)\mathrm{d}B_t \geq \sum_{j=1}^{\infty} 1 = \infty$ diverges.

Comment: if we add restrictions beyond L^2, such as smoothness, then we can make these integrals defined simultaneously. The pathology above comes from taking a very high-frequency function f which correlates with the unusually high frequencies of B_t. In fact the very interesting *rough path theory* gives an alternate theory of stochastic integrals that are defined for all integrands simultaneously.

2.4 Illustrative Examples of Stochastic Integration

Question: What is $\int_0^1 B_t \mathrm{d}B_t$?
In classical calculus, one may use integration by parts to observe that $\int_0^1 f(t)\mathrm{d}f(t) = \frac{f(1)^2 - f(0)^2}{2}$. Applying this to our problem, we'd get that $\int_0^1 B_t\mathrm{d}B_t = \frac{B_1^2 - B_0^2}{2} = \frac{B_1^2}{2}$.
To observe why this is *doesn't work*, recall that $s \mapsto \int_0^s \sigma(t)dB_t$ was supposed to be a martingale, but $\mathbb{E}\left[\frac{B_1^2}{2}\right] \neq 0$ which is impossible for a martingale. Thus, traditional calculus fails us here.

Discrete Approximation. We can construct a discrete approximation of this integral with size ϵ time steps. The correct way to do so is utilizing the left endpoints of the discretization intervals.
Left endpoint approximation of $\int_0^1 B_t\mathrm{d}B_t$ is $B_0(B_\epsilon - B_0) + B_\epsilon(B_{2\epsilon} - B_\epsilon) + B_{2\epsilon}(B_{3\epsilon} - B_{2\epsilon})\dots$
We have, for now, a discrete-time martingale: $\mathbb{E}[B_{k\epsilon}(B_{(k+1)\epsilon} - B_{k\epsilon})|\mathcal{F}_{k\epsilon}] = 0$.

If we instead tried *averaging* left and right endpoints, we would obtain the following telescoping sum:

$$\frac{B_\epsilon + B_0}{2}(B_\epsilon - B_0) + \frac{B_{2\epsilon} + B_\epsilon}{2}(B_{2\epsilon} - B_\epsilon) + \dots$$
$$= \frac{1}{2}((B_\epsilon^2 - B_0^2) + (B_{2\epsilon}^2 - B_\epsilon^2) + \dots)$$

$$= \frac{B_1^2}{2}$$

Thus, it is important to use the left endpoint for discrete approximation.

Restrictions on [0,1] partition. We must also have that the partition of $[0,1]$ used during discretization must be independent of B_t. This was true in our earlier demonstration, with $0 < \epsilon < 2\epsilon < \ldots < 1$.
To see why this is true, consider $\{t_i\}$, with $0 = t_0 < t_1 < \ldots < t_n = 1$. Our left-approximation is

$$B_0(B_{t_1} - B_0) + B_{t_1}(B_{t_2} - B_{t_1}) + \ldots = \frac{B_1^2}{2} - \frac{1}{2}\left(\sum_{k=0}^{n-1}(B_{t_{k+1}} - B_{t_k})^2\right).$$

Typically, $(B_{t_{k+1}} - B_{t_k})^2 \approx t_{k+1} - t_k$, but not in the worst case (where t_k depends on BM).
Question: For which processes σ_t can we integrate $\mathrm{d}B_t$? (Assuming we have $(B_t, \mathscr{F}_t)$).
We initially attempt $\mathbb{E}\int_0^1 \sigma_t^2 \mathrm{d}t < \infty$, where σ_t is $\mathscr{F}_t$-measurable. However this allows for the possibility that $\sigma_t = F(t)$ a.s. for a deterministic F which is not Lebesgue measurable as a function on $[0,1]$. So we need to require slightly more.

We will utilize the notion of *progressively measurable processes*.

3 Progressively Measurable Processes

Definition. $\{\sigma(t,\omega) : t \geq 0, \omega \in \Omega\}$ is *progressively measurable* if $\forall t \geq 0$, $\sigma : [0,t] \times \Omega \to \mathbb{R}$ is measurable with respect to $\mathrm{Borel}([0,t]) \otimes \mathscr{F}_t$.

Claim. If $\sigma(t,\omega)$ is adapted to $\mathscr{F}_t$ and is either left continuous or right continuous in t, it is also progressively measurable.
Proof. In this proof, we will approximate σ by piecewise constant functions σ_n, and claim σ as the limit of such progressively measurable functions σ_n.
Assume right continuity. We will focus on $t \in [0,1]$.
At stage n, take steps of size $\frac{1}{2^n}$. Define $\sigma_n(s,\omega) = \sigma\left(\frac{k+1}{2^n}, \omega\right)$ for $s \in \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]$
Sub-claim 1: Each σ_n is progressively measurable. (exercise to reader)
Sub-claim 2: $\sigma_n(s,\omega) \xrightarrow{n\to\infty} \sigma(s,\omega)\ \forall s, \omega$.
By right continuity, $\forall s$, $\lim_{\epsilon \downarrow 0} \sigma(s+\epsilon, \omega) = \sigma(s,\omega)$.
Consider $\epsilon_n = \frac{k+1}{2^n} - s$, where $\epsilon_n \in [0, 2^{-n})$. Then we know that $\epsilon_n \downarrow 0$. Therefore,

$$\lim_{n\to\infty} \sigma_n(s,\omega) = \lim_{n\to\infty} \sigma\left(\frac{k+1}{2^n}, \omega\right) = \lim_{\epsilon\downarrow 0} \sigma(s+\epsilon, \omega) = \sigma(s,\omega).$$

The limit of progressively measurable functions is progressively measurable. Therefore $\sigma(t,\omega)$ is progressively measurable, concluding the proof. □

4 Itô Isometry

Let's define Itô integration for general progressively measurable σ.
To do so, we will define it for piecewise constant σ, and take the L^2 limit.
Step 1. Fix $0 = t_0 \leq t_1 \leq \ldots \leq t_k = 1$.
Define $H(t,\omega) = \sum_{i=0}^k A_i(\omega) \mathbb{1}_{t \in [t_i, t_{i+1}]}$, where A_i are $\mathscr{F}_t$-measurable functions. Such H are called *simple functions*.
For simple function H, $\int_0^1 H(t,\omega)\mathrm{d}B_t = \sum_{i=0}^k A_i \times (B_{t_{i+1}} - B_{t_i})$.

Claim (L^2 isometry). If H is a simple function and $\mathbb{E}\int_0^\infty H(s)^2 \mathrm{d}s < \infty$, then

$$\mathbb{E}\left[\left(\int_0^1 H(s)\mathrm{d}B_s\right)^2\right] = \mathbb{E}\int_0^1 H(s)^2 \mathrm{d}s.$$

Proof. We first decompose

$$\mathbb{E}\left[\left(\int_0^1 H(s)\mathrm{d}B_s\right)^2\right] = \underbrace{\mathbb{E}[\sum_i A_i^2 (B_{t_{i+1}} - B_{t_i})^2]}_{\text{Term 1}} + \underbrace{2\sum_{i<j} \mathbb{E}[A_i A_j (B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j})]]}_{\text{Term 2}}.$$

Term 1:

$$\mathbb{E}[\sum_i A_i^2 (B_{t_{i+1}} - B_{t_i})^2] = \mathbb{E}[\sum_i A_i^2 (t_{i+1} - t_i)] = \mathbb{E}\int_0^1 H(s)^2 \mathrm{d}s.$$

Term 2:

$$2\sum_{i<j} \mathbb{E}[A_i A_j (B_{t_{i+1}} - B_{t_i})(B_{t_{j+1}} - B_{t_j})]] = 0$$

because

$$\mathbb{E}[B_{t_{j+1}} - B_{t_j} | \mathscr{F}_{t_j}] = 0.$$

All other factors are $\mathscr{F}_{t_j}$-measurable.
Therefore,

$$\mathbb{E}\left[\left(\int_0^1 H(s)\mathrm{d}B_s\right)^2\right] = \mathbb{E}\int_0^1 H(s)^2 \mathrm{d}s. \square$$