Statistics 212: Lecture 16 (April 2, 2025)

Construction of Ito Integral

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1 Constructing the Itô Integral

Goal: Define

$$\int_0^\infty H(t)\,dB(t),$$

where $H(t) = H(t,\omega)$ is a progressively measurable process. By definition, this means we require that for each $T > 0$,

$$H\colon [0,T]\times\Omega \to \mathbb{R}$$

is measurable with respect to $\mathscr{B}([0,T])\otimes F_T$. We also require the square-integrability:

$$\mathbb{E}\left[\int_0^\infty H(t,\omega)^2\,dt\right] < \infty.$$

(Note that since integration and expectation are formally equivalent, this just saying $H(t,\omega)$ is in L^2 on the product space it is defined on.)

1.1 Definition on simple processes.

A *simple process* $H(t,\omega)$ has the form

$$H(t,\omega) = \sum_{i=1}^{k} A_i(\omega)\,\mathbf{1}_{(t_i,\,t_{i+1}]}(t),$$

for a partition $0 = t_0 < t_1 < \cdots < t_k = T$, where each $A_i(\omega)$ is F_{t_i}-measurable. Then we define

$$\int_0^\infty H(t,\omega)\,dB_t = \sum_{i=1}^{k} A_i(\omega)\left[B_{t_{i+1}} - B_{t_i}\right].$$

Key property: Itô $\mathscr{L}^2$ isometry.
On simple processes, we have

$$\mathbb{E}\left[\left(\int_0^T H(t,\omega)\,dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T H(t,\omega)^2\,dt\right].$$

1.2 Approximation by Simple Processes

Fact 16.1 Let H be a progressively measurable process such that

$$\mathbb{E}\Big[\int_0^\infty H(t,\omega)^2\,dt\Big] < \infty.$$

Then there exists a sequence of simple processes $(H_n)_{n\geq 1}$ such that

$$\mathbb{E}\Big[\int_0^\infty \big(H_n(t) - H(t)\big)^2\,dt\Big] \longrightarrow 0 \quad \text{as } n \to \infty.$$

Proof (outline):

(a) *Truncating the time domain.*
Choose T large enough so that

$$\mathbb{E}\Big[\int_T^\infty H(t,\omega)^2\,dt\Big] \le \varepsilon.$$

Then it suffices to approximate $H(t,\omega)\,\mathbf{1}_{[0,T]}(t)$, which we denote by $H^{(1)}(t,\omega)$.

(b) *Clipping the range.*
Define the *clip* function

$$\mathrm{Clip}_N(x) = \min\big(N, \max(-N, x)\big).$$

Choose N large enough so that

$$\int_0^T \big[H^{(1)}(t,\omega) - \mathrm{Clip}_N\big(H^{(1)}(t,\omega)\big)\big]^2\,dt \le \varepsilon.$$

Set $H^{(2)}(t,\omega) = \mathrm{Clip}_N\big(H^{(1)}(t,\omega)\big)$. Existence of such T and N follows from the dominated convergence theorem.

(c) *Step-function approximation in time.*
Partition $[0,T]$ using $t_k = \frac{k}{2^n}$ for $k = 0,1,2,\ldots,2^n T$. Then define

$$H_n^{(3)}(t,\omega) = 2^n \int_{t_k}^{t_{k+1}} H^{(2)}(s,\omega)\,ds \quad \text{for } t \in [t_{k+1}, t_{k+2}).$$

Each $H_n^{(3)}$ is again progressively measurable (adapted and right-continuous) (as long as $H^{(2)}$ is progressively measurable).

By the Lebesgue differentiation theorem, or the following martingale argument, one may show $H_n^{(3)} \to H^{(2)}$ in L^2, as $n \to \infty$.

Define the filtration

$$\mathscr{G}_n = \sigma\Big(\big\{[0,2^{-n}], [2^{-n}, 2\cdot 2^{-n}], \ldots\big\}, \mathscr{F}\Big).$$

Then

$$\mathbb{E}\big[\,H(t,\omega)\,\big|\,\mathscr{G}_n\big] = 2^n \int_{t_k}^{t_{k+1}} H(t,\omega)\,dt \quad \text{for } t \in [t_k, t_{k+1}].$$

Define the *Doob martingale* $(M_n)_{n\geq 1}$ by

$$M_n = \mathbb{E}\big[\,H\,\big|\,\mathscr{G}_n\big] \quad \text{for } H \in \mathscr{L}^2.$$

Then

$$M_n \xrightarrow{\text{a.s.}, \mathscr{L}^2} \mathbb{E}\big[H\,\big|\,(\mathscr{G}_1, \mathscr{G}_2, \ldots)\big],$$

where the latter is the conditional expectation w.r.t. $\mathscr{B}(\mathbb{R}) \otimes F$

1.3 The Itô integral

(i) *Setup.* Let H be progressively measurable with $\mathbb{E}\Big[\int_0^\infty H(t,\omega)^2\,dt\Big] < \infty$. Take a sequence of simple processes $(H_n)_{n\geq 1}$ such that $\mathbb{E}\Big[\int_0^\infty (H-H_n)^2\,dt\Big] \to 0$.

(ii) *Definition.* Define

$$\int_0^\infty H(t)\,dB_t \;=\; \lim_{n\to\infty}\int_0^\infty H_n(t)\,dB_t \quad \text{(limit in } \mathscr{L}^2\text{)}.$$

(a) *Existence.* Show that $\big\{\int_0^\infty H_n\,dB_t\big\}_n$ is Cauchy in L^2.

(b) *Uniqueness.* The limit cannot depend on the particular approximating sequence $H_n \to H$.

Hence:

(i) $\int_0^\infty H_n\,dB_t \to \int_0^\infty H\,dB_t$ in L^2,

(ii) The limit is the same for any other sequence $\widetilde{H}_n \to H$,

(iii) $\mathbb{E}\big[\big(\int_0^\infty H\,dB_t\big)^2\big] = \mathbb{E}\big[\int_0^\infty H(t)^2\,dt\big]$.

(i) Apply L^2 isometry.
We have

$$\mathbb{E}\Big[\Big(\int_0^\infty H_n dB_t - \int_0^\infty H_m dB_t\Big)^2\Big] \;=\; \mathbb{E}\Big[\int_0^\infty (H_n - H_m)^2\,dt\Big] \xrightarrow{n,m\to\infty} 0.$$

Thus $\{\int_0^\infty H_n\,dB_t\}$ is Cauchy in $\mathscr{L}^2$ and hence convergent in $\mathscr{L}^2$.

(ii) Uniqueness via interleaving.
If $\{H_n\} \to H$ in L^2 and $\{\widetilde{H}_n\} \to H$ in L^2, then the interleaved sequence $H_1, \widetilde{H}_1, H_2, \widetilde{H}_2\cdots$ also converges to H. Their Itô integrals must converge to the same limit, ensuring uniqueness of the definition $\int_0^\infty H\,dB_t$.

(iii) $\mathscr{L}^2$ Isometry:
By construction

$$\int_0^\infty H\,dB_t \;:=\; \lim_{n\to\infty}\int_0^\infty H_n\,dB_t \quad (\text{in } L^2).$$

Hence the L^2 norms converge, establishing the isometry property.

2 Time progressive Itô integral

We set

$$\int_0^t H(s)\,dB_s \;=\; \int_0^\infty H(s)\,\mathbf{1}_{[0,t]}(s)\,dB_s,$$

allowing us to discuss the structure of the stochastic process $M(t) = \int_0^t H(s)\,dB_s$ as t varies.

Key tool: L^2 maximal inequality.

Theorem 16.1 Let H be progressively measurable with $\mathbb{E}\Big[\int_0^\infty H(s)^2\,ds\Big] < \infty$
Then there exists an *almost surely continuous modification* of $\big\{M_t = \int_0^t H(s)\,dB_s : t \geq 0\big\}$ which is also a martingale.
Modification: Two stochastic processes X_t and X'_t are said to be *modifications* of each other if for all t, $\mathbb{P}\big(X_t \neq X'_t\big) = 0$

Proof:

Case: Simple processes
If $H(s,\omega)$ is *simple*, say

$$H(s,\omega) = \sum_{i=1}^{k} A_i(\omega)\, \mathbf{1}_{(t_i,t_{i+1}]}(s),$$

then

$$\int_0^t H(s)\, dB_s = \sum_{i=1}^{k} A_i \left(B_{t_{i+1}\wedge t} - B_{t_i\wedge t}\right),$$

which is

- a.s. continuous (as BM is a.s. continuous)
- a martingale
- progressively measurable

Case: General H
Approximate $H_n \xrightarrow{L^2} H$, define

$$M_n(t) = \int_0^t H_n(s)\, dB_s,$$

By Doob's L^2 Maximal Inequality

$$\mathbb{E}\left[\left(\sup_{0\le t<\infty}\left|M_n(t) - M_{n'}(t)\right|\right)^2\right] \le 4\,\mathbb{E}\left[(M_n(\infty) - M_{n'}(\infty))^2\right] = 4\mathbb{E}\left[\int_0^\infty (H_n(s) - H_{n'}(s))^2\, ds\right] \to 0$$

Fix $1 < \alpha < 2$

By Chebyshev,

$$\mathbb{P}\left(\sup_{t\ge 0}|M_n(t) - M_{n'}(t)| > \alpha^{-k}\right) \le 4\,\alpha^{2k}\mathbb{E}\left[\int_0^\infty (H_n(s) - H_{n'}(s))^2\, ds\right] = \left(\frac{\alpha^2}{4}\right)^k \text{ (for } n, n' > n_k \text{ large).}$$

It follows that $\{M_{n_k} : k \ge 0\}$is a.s. Cauchy in $C\left([0,\infty), d_{sup}\right)$ for suitable subsequence $n_1 < n_2 < \cdots$

Because we may choose $\{n_k\}$ so that

$$\mathbb{P}\left[\sup_{t\ge 0}|M_{n_{k+1}}(t) - M_{n_k}(t)| \ge \alpha^{-k}\right] \le \left(\frac{\alpha^2}{4}\right)^k$$

which is summable.

Hence by Borel-Cantelli Lemma, all these events hold past random $\overline{k} = \overline{k}(\omega)$. If $k, k' > \overline{k}$,

$$\sup_t |M_{n_k}(t) - M_{n_{k'}}(t)| \le \sum_{j\ge\min(k,k')} \alpha^{-j} \to 0$$

as $k, k' \to \infty$

Thus, M_{n_k} converge a.s. to an a.s. continuous limit $M'(t)$, which is a modification of $M(t) = \int_0^t H(s)\, dB_s$.