Statistics 212: Lecture 18 (April 9, 2025)

Existence of solutions to SDEs, Picard iteration

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Last time, we talked about Ito's formula. We mentioned an extension called Tanaka's formula that can be used for $|B_t|$, where the Ito correction at 0 is called Brownian local time. This sounds fancy but is simply the scaling limit of the following. Think of SRW, $X_i\, i.i.d. \sim \pm 1, S_n = X_1 + \cdots + X_n$. Then

$$|S_n| - \sum_{k=0}^{n-1} \mathbf{1}_{S_k=0} \quad \text{is a martingale.}$$

In the scaling limit, $|S_n| \to |B_t|$ and the second term is the occupation density of BM (local time) at 0.

1 SDEs

Example. Let (μ, σ) be constants.
Solve:

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t, \quad Y_0 = 1$$

Generically, solutions will take the form $Y_t = f(t, B_{[0,t]})$ of a function depending on t and the full history Brownian Motion up to time t. Here, we will guess that Y_t is of the simpler form $f(t, B_t)$ for a nice function f (and it turns out we will succeed). From Ito's formula, we get

$$\begin{aligned} dY_t &= d_x f(t, B_t) dB_t + \left(\frac{d_{xx} f(t, B_t)}{2} + d_t f(t, B_t) \right) dt \\ &\stackrel{?}{=} \sigma f(t, B_t) dB_t + \mu f(t, B_t) dt \end{aligned}$$

To check if they are equal, we can compare the terms. Let's start with the dB_t term.

$$\begin{aligned} &d_x f = \sigma f \\ &\implies f(t, x) = A(t) e^{\sigma x} \end{aligned}$$

This is just a function that solves the equation for this term. It will become clear in the end why this is a unique solution. Now, we check the dt term

$$\begin{aligned} &\frac{\sigma^2 A(t)}{2} + A'(t) = \mu A(t) \\ &\implies A(t) = C e^{(\mu - \frac{\sigma^2}{2})t}, \quad C = Y_0 = 1 \end{aligned}$$

Question 1.1. *Is Y_t unique? When should solutions exists?*

Consider SDE

$$dX_t = a(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = 0 \tag{1}$$

Assume a, σ are uniformly Lipschitz:

$$|a(x) - a(y)| \leq A|x - y|$$
$$|\sigma(x) - \sigma(y)| \leq A|x - y|$$

Theorem 1.2. *The SDE* (1) *has a unique solution.*

More generally, the same will hold (with similar proof) for coefficients $a(t, X_{[0,t]}), \sigma(t, \tilde{X}_{[0,t]})$ obeying the Lipschitz-in-path-space condition

$$|a(t, X_{[0,t]}) - a(t, \tilde{X}_{[0,t]})| \leq A \sup_{s\leq t} |X_s - \tilde{X}_s|.$$

To prove the existence of the solution, we can use Picard iteration.

Note that (by taking $\sigma_t = 0$) the above result generalizes the standard result on ordinary differential equations.

1.1 Picard iteration

Define a sequence

$$\left\{X_t^k, t \in [0, T]\right\}_{k=0}^{\infty} \in C([0, T])$$

and we will construct it to be Cauchy

$$\begin{aligned} X_t^0 &= 0 \quad \forall t \\ X_t^k &= \int_0^t a(X_s^{k-1})dt + \int_0^t \sigma(X_s^{k-1})dB_s \end{aligned} \tag{2}$$

The second integral is defined, because the integrand is progressively measurable as X_s only depends on the Brownian Motion up to time s. Additionally, one can show by induction that

$$E|X_t^k| \leq C(A, T, k) \quad \forall t \leq T$$

We will now prove that this is a Cauchy sequence, so that the limit exists. Further, the limit will be a fixed point of (2), i.e. a solution of the SDE. As a first technical step, we will need to use Doob's L^2 maximal inequality to reduce to fixed-time L^2 distances (rather than the more subtle uniform norm $d_{\sup}$). After that, the idea will be to show that integration brings things closer together similarly to the ODE setting. (You are encouraged to look up/review the proof of existence/uniqueness for ODE solutions to help understand the strategy below.)

$$\begin{aligned} X_t^{k+1} - X_t^k = &\int_0^t a(X_s^k) - a(X_s^{k-1})dt \\ &+ \int_0^t \sigma(X_s^k) - \sigma(X_s^{k-1})dB_s = Y_t^k + Z_t^k \end{aligned}$$

We want the functions to be Cauchy in pathspace

$$\begin{aligned} &\Pr(\max_{0\leq t\leq T} |X_t^{k+1} - X_t^k| > \epsilon) \\ &\leq \Pr\left(\int_0^T |a(X_t^k) - a(X_t^{k-1})|dt \geq \frac{\epsilon}{2}\right) + \Pr\left(\max_{t\leq T} Z_t^k \geq \frac{\epsilon}{2}\right) \end{aligned}$$

$$\le \frac{4}{\epsilon^2} E\left[\left(\int_0^T |a(X_t^k) - a(X_t^{k-1})| dt\right)^2\right] + \frac{16}{\epsilon^2} E(|Z_T^k|^2)$$

Where the first and second term come from the application of the Chebyshev and the L^2 maximal inequality. We can now apply Cauchy-Schwartz and the Ito isometry and get

$$\le \frac{4TA^2}{\epsilon^2} E\int_0^T \left(X_t^k - X_t^{k-1}\right) dt + \frac{16A^2}{\epsilon^2} E\int_0^T (X_t^k - X_t^{k-1})^2 dt \tag{3}$$

$$\le \frac{16}{\epsilon^2} A^2(1+T) E\int_0^t (X_t^k - X_t^{k-1})^2 dt \tag{4}$$

Now we only need to bound an integral in time, not a maximum anymore. Now we use the inequality $(a+b)^2 \le 2a^2 + 2b^2$ to decompose the deterministic and stochastic part of the integral. Look at the L^2 difference at a given time:

$$\begin{aligned} E(X_t^{k+1} - X_t^k)^2 &\le 2E\left[\left(\int_0^t a(X_s^k) - a(X_s^{k-1}) ds\right)^2 + \left(\int_0^t \sigma(X_s^k) - \sigma(X_s^{k-1}) dB_s\right)^2\right] \\ &\le 2A^2(t+1)\int_0^t E(X_s^k - X_s^{k-1})^2 ds \end{aligned}$$

The last inequality results from the application of the Lipschitz condition and Cauchy-Schwarz. The t before the integral comes from the dt term and the 1 comes from the dB_t term. We can see that we bounded that expectation by itself in an earlier time. Let

$$f_t^k = E(X_t^{k+1} - X_t^k)^2$$

We just showed that

$$\begin{aligned} f_t^k &\le 2A^2(t+1)\int_0^t f_s^{k-1} ds \\ &\le \cdots \le (2A^2(t+1))^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} f_{t_k}^0 d_{t_k} \cdots d_{t_1} \end{aligned}$$

with

$$\begin{aligned} f_s^0 &= E(X_t^1 - X_t^0)^2, \quad X_s^0 \equiv 0 \\ X_s^1 &= \int_0^s a(0) dr + \int_0^s \sigma(0) dB_r, \implies f_s^0 \le O(T) \end{aligned}$$

Together with

$$\int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} 1 dt_k \cdots d_{t_1} = \frac{t^k}{k!}$$

we can bound our previous result

$$(2A^2(t+1))^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} f_{t_k}^0 d_{t_k} \cdots d_{t_1} \le \frac{O(T)^k}{k!}$$

Now we can bound (4), and let ϵ be dependent on k

$$\frac{16}{\epsilon^2} A^2(1+T) E\int_0^t (X_t^k - X_t^{k-1})^2 dt \le \frac{O(T)^{O(k)}}{\epsilon^2 k!}$$

Finally, to show that the sequence is Cauchy, we choose the sequence ϵ_k such that

$$\sum_{k=1} \epsilon_k < \infty,$$

$$\sum_k \epsilon_k^{-2} \frac{O(T)^{O(k)}}{k!} < \infty.$$

For example, we can take $\epsilon_k = (k!)^{-\frac{1}{3}}$.

We have now shown that $X^k \to X^\infty$ almost surely in $d_{\text{sup}}([0,T])$. It follows that

$$\int_0^t a(X_s^k)ds \to \int_0^t a(X_s^\infty)ds$$

uniformly in $d_{\text{sup}}([0,T])$ as well. Further, combining Doob's maximal inequality with the Ito isometry shows that

$$\int_0^t \sigma(X_s^k)dB_s \to \int_0^t \sigma(X_s^\infty)dB_s$$

in probability in $C([0,T])$ (wrt d_{sup}). Since by definition

$$X_t^{k+1} = \int_0^t a(X_s^k)ds + \int_0^t \sigma(X_s^k)dB_s$$

it follows that the in-probability limits in $C([0,T])$ (wrt) d_{sup}) also agree, i.e.

$$X_t^\infty = \int_0^t a(X_s^\infty)ds + \int_0^t \sigma(X_s^\infty)dB_s.$$

1.2 Preview of next time:

We have shown existence. For uniqueness, suppose:

$$X_t = \int_0^t a(X_s)ds + \int_0^t \sigma(X_s)dB_s$$
$$Y_t = \int_0^t a(Y_s)ds + \int_0^t \sigma(Y_s)dB_s$$

Then we will examine

$$E\left[(X_t - Y_t)^2\right]$$

and use Ito's formula and Gronwall's inequality to obtain uniqueness.

We will also see that time-discretization gives a valid approximation under the same Lipschitz assumptions, and see SDEs with more subtle solution theory.