Statistics 212: Lecture 19 (April 14, 2025)

Stochastic Differential Equations II

Instructor: Mark Sellke

Scribe: Kevin Liu

1 Uniqueness of SDE Solutions

Typical set-up: we have a stochastic differential equation

$$dX_t = \sigma(X_t)dB_t + v_t dt$$

where σ, v are Lipschitz from $\mathbb{R}$ to $\mathbb{R}$. Last time, we showed the existence of a stochastic process that satisfies this SDE using Picard iteration. To demonstrate uniqueness, we can suppose that X_t and Y_t are solutions to this SDE. Consider $Z_t = X_t - Y_t$. We have

$$Z_t = X_t - Y_t = \int_0^t [\sigma(X_s) - \sigma(Y_s)]dB_s + \int_0^t [v(X_s) - v(Y_s)]ds,$$

and note that $|v(X_s) - v(Y_s)| \le L|Z_t|$ from some Lipschitz bounding. Applying Ito's formula to Z_t^2, we also get

$$d(Z_t^2) = 2Z_t(\sigma(X_t) - \sigma(Y_t))dB_t + (\sigma(X_t) - \sigma(Y_t))^2 dt + 2Z_t(v(X_s) - v(Y_s))dt,$$

implying that

$$E[Z_t^2] \le L\int_0^t E[Z_s^2]ds.$$

If we let $f(t) = \int_0^t E[Z_s^2]ds$, this implies $f'(t) \le CLf(t)$ by some Lipschitz bounding, which further implies $f(t)e^{-CLt}$ is decreasing with $f(0) = 0$. Thus, $f(t) = 0$ for all t. At any given time, we see $X_t = Y_t$ almost surely.

Definition 1.1 (Locally Lipschitz)**.** We call σ locally Lipschitz if $|\sigma(x) - \sigma(y)| \le L(R)|x - y|$ if $|x|, |y| \le R$.

Question 1.2. *We assumed that σ, v are globally Lipschitz. What if they are locally Lipschitz, e.g., $dX_t = X_t^2 dB_t + X_t^3 dt$?*

Existence and uniqueness will hold until an "explosion time." For motivation behind this, let's take an ODE $dX_t = X_t^2 dt$. A solution to this is $X_t = \frac{1}{1-t}$, and this process will blow up when approaching $t = 1$. For any $R > 0$, we can modify coefficients outside $B_R(0)$, the ball centered at 0 with radius R. We can construct the following functions:

$$\sigma^R(x) = \begin{cases} \sigma(x), & |x| \le R \\ \sigma(Rx/|x|), & |x| \ge R \end{cases}$$

$$v^R(x) = \begin{cases} v(x), & |x| \le R \\ v(Rx/|x|), & |x| \ge R. \end{cases}$$

Note that σ^r, v^r are globally Lipschitz, and define τ^R to be the first time $|X_t| = R$. Then $\lim_{R\to\infty} \tau^R = \tau$, which is the "explosion time."

2 Another characterization of BM

Definition 2.1. X_t is a "strong solution" to a SDE if it is adapted to the filtration by the driving Brownian motion B_t, i.e., $\mathscr{F}_t^X \subseteq \mathscr{F}_t^B$. X_t is a "weak solution" if it is adapted wrt $\mathscr{F}_t \supseteq \mathscr{F}_t^B$ and B_t is BM with respect to $\mathscr{F}_t$.

Example. $dX_t = \text{sign}(X)dB_t$ where $\text{sign}(x) = 1$ for $x \ge 0$ and -1 otherwise.

This is an example of a SDE that has a weak solution but not a strong solution. By inspection, X_t should be a Brownian motion of some kind. So let X_t be BM, and $B_t = \int_0^t \text{sign}(X_s)dX_s$ (we'll show that this integral is a Brownian motion later). Since $\text{sign}(X_s)^2 = 1$, we see that this yields a solution. Define $\mathscr{F}_t^X = \sigma(X_s : s \le t)$ and $\mathscr{F}_t^B = \sigma(B_s : s \le t)$. It turns out

$$\mathscr{F}_t^{|X|} = \mathscr{F}_t^B \subsetneq \mathscr{F}_t^X.$$

Some intuition behind this: if we look at $\tilde{X}_t = -X_t$ for all t, this yields the same B_t. The existence of $\tilde{X}_t$ means we don't have uniqueness of solutions, and our solution isn't even measurable with respect to the filtration generated by our Brownian motion. We see $\int_0^t \text{sign}(X_s)dB_s$ is defined for any nice filtration $\mathscr{F}$ such that B_t is a Brownian motion with respect to $\mathscr{F}$. For this example, we have to enlargen our state space to find a solution, so we have $\mathscr{F}_t \supsetneq \mathscr{F}_t^B$.

Theorem 2.2 (Levy's Characterization of BM)**.** *BM is the unique continuous martingale such that $B_t^2 - t$ is also a martingale. In other words, any such process that satifies the above characteristics has the law of BM on $C([0,1])$.*

Question 2.3. *Is $B_t = \int_0^t \text{sign}(X_s)dX_s$ BM?*

Question 2.4. *Can we show any weak solution X_t is BM?*

Yes! Both questions can be shown with the above theorem.

Proof. We cover two proofs to Levy's characterization of BM.

1. You can define Ito integration with respect to any continuous martingale X_t. You need a continuous increasing process A_t such that $X_t^2 - A_t$ is a martingale. (Non-trivial fact that you can always construct an A_t given that X_t is a martingale). For example, if $X_t = \int_0^t \sigma_s dB_s$, then $A_t = \int_0^t \sigma_s^2 ds$.

   Generalized Ito's lemma:
   $$f(X_t) = \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)dA_s.$$
   Working with characteristic functions, let $Y_t = f(X_t) = \exp(i\omega X_t)$. If X_t and $X_t^2 - t$ are martingales, then Ito results in
   $$Y_t - Y_0 = \int_0^t i\omega Y_s dX_s - \frac{\omega^2}{2}\int_0^t Y_s ds.$$
   We have
   $$E[Y_t] = 1 - \frac{\omega^2}{2}\int_0^t E[Y_s]ds \implies E[Y_t] = \exp(-\omega^2 t/2),$$
   meaning that $X_t \sim \mathcal{N}(0,t)$.

2. Proving martingale CLT in discrete time. The idea is start with X_t and discretize time into buckets, resulting in a discrete-time martingale. Somehow using CLT to get a Gaussian from this martingale given our understanding of the variance of each bucket.

   Theorem 2.5 (Levy). *Suppose that we have a triangular array $\{\xi_{n,i}\}_{i\leq m(n)}$ with $E[\xi_{n,i}|\mathscr{F}_{n,i-1}] = 0$ and $\xi_{n,i}$ adapted to $\{\mathscr{F}_{n,i}\}_{i\leq m(n)}$. Define $S_{n,k} = \sum_{i=1}^{k} \xi_{n,i}$ and $V_{n,k}^2 = \sum_{i=1}^{k} E[\xi_{n,i}^2|\mathscr{F}_{n,i-1}]$ (defining the variance as σ_i^2. Assume $V_{n,m(n)}^2 = 1$. Recall the Lindeberg condition which states for all δ, we have $\lim_{n\to\infty}\sum_{i=1}^{m(n)} E[\xi_{n,i}^2 \mathbf{1}_{|\xi_{n,i}|>\delta}] = 0$.*

   Then $S_{n,m(n)} \xrightarrow{d} \mathcal{N}(0,1)$.

   For our use case here, we can let $\xi_{n,i} = X_{i/n} - X_{(i-1)/n}$ and $m(n) = n$. So $S_{n,n} = X_1$. By continuity, the Lindeberg condition follows (choosing n large enough for every δ). Thus, $X_1 \sim \mathcal{N}(0,1)$.

   □

Next time, we'll go through a proof of this martingale CLT using the Lindeberg replacement method!