Statistics 212: Lecture 21 (April 21, 2025)

Brownian Motion and Complex Analysis

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1 Recap Martingale CLT

For full proof see Lecture 20 notes.

Theorem 1.1. *Let $\{X_{n,i}, 1 \le i \le m(n)\}$ be a triangular array for $n \in \mathbb{Z}_+$. In our application $m(n) = n$. Define:*

$$S_{n,k} = \sum_{i=1}^{k} X_{n,i}, \quad V_{n,k}^2 = \sum_{i=1}^{k} \mathbb{E}[X_{n,i}^2 \mid \mathscr{F}_{n,i-1}]$$

Assume:

- *(Martingale difference)* $\mathbb{E}[X_{n,i} \mid \mathscr{F}_{n,i-1}] = 0$.
- $V_{n,m(n)}^2 = 1$.
- *(Lindeberg condition)*

$$\forall \delta > 0, \quad \lim_{n\to\infty} \sum_{i=1}^{m(n)} \mathbb{E}\left[X_{n,i}^2 \cdot \mathbf{1}_{\{|X_{n,i}| > \delta\}}\right] = 0.$$

Then, $S_{n,m(n)} \xrightarrow{d} \mathcal{N}(0,1)$.

Proof Sketch We had a martingale difference sequence $X_1, \dots, X_{n(m)}$ and we define $Y_i = \sigma_i g_i$ for $g_i \sim \mathcal{N}(0,1)$ and $\sigma_i^2 = \mathbb{E}[X_i^2 | \mathscr{F}_{i-1}]$. We want to bound (for smooth $f : \mathbb{R} \to \mathbb{R}$) the following sum

$$\sum_{k=1}^{m(n)} |A_k| = \sum_{k=1}^{m(n)} |\mathbb{E}[f(X_1 + \cdots + X_k + Y_{k+1} + \cdots + Y_m) - f(X_1 + \cdots + Y_k + Y_{k+1} + \cdots + Y_m)]|$$

The trick is that conditionally on $\mathscr{F}_{k-1}$, $(Y_{k+1} + \cdots + Y_m)$ is gaussian with known variance. The idea of the proof is to use backward induction. If you've determined what σ_1 is then you know that the future part is an independent centered Gaussian. The idea is to start by looking at the last variance term and work backwards, applying our inductive hypothesis. One way to get started is by recalling that $\sum_{j=1}^{m(n)} \sigma_j^2 = 1$ holds almost surely. Thus, $\sum_{j=k+1}^{m} \sigma_j^2 = 1 - \sum_{j=1}^{k} \sigma_j^2 = 1 - \sum_{j=1}^{k} \mathbb{E}[X_j^2 | \mathscr{F}_{j-1}]$ is $\mathscr{F}_{k-1}$-measurable.

Assuming the above trick, we now move on to define

$$f_k(X) = \mathbb{E}[f(X + g \cdot \sigma_{k+1:m})]$$

where $\sigma_{k+1:m} = \sum_{j=k+1}^{m} \sigma_j^2$. We also know that f_k is $\mathscr{F}_{k-1}$ measurable and by properties of convolutions, we have that

$$|f_k''|_\infty \le |f''|_\infty$$

Then, applying the law of iterated expectation and Taylor expanding f_k, we have that

$$\begin{aligned} |A_k| &\le |\mathbb{E}[\mathbb{E}[f(X_1+\cdots+X_k+Y_{k+1}+\cdots+Y_m) - f(X_1+\cdots+Y_k+Y_{k+1}+\cdots+Y_m)|\mathscr{F}_{k-1}]]| \\ &\le C(f)\cdot\mathbb{E}[\epsilon X_k^2 + CX_k^2\cdot I_{|X_k|>\delta} + |Y_k|^3] \end{aligned}$$

Where the first term in the expectation is small by total variance and the second is small by Lindeberg. The $|Y_k|^3$ term is a bit more tricky to control, but what was done so by Prof Sellke in Section 2 of the Lecture 20 notes.

2 Complex Analysis and Brownian Motion

2.1 Planar Brownian Motion

Warmup: Planar Brownian Motion is not recurrent but is *neighborhood recurrent* meaning that $\sup_{t\ge 0}\{t : |B_t| \le \epsilon\} = \infty$ almost surely $\forall \epsilon > 0$.
Remark: This may be surprising because a random walk in 2-D is recurrent and planar BM is the scaling limit of that, but 2 dimensions is actually the borderline case.

Proof. Let $X_t = \log|\vec{B}_t|^2 = \log((B_t^x)^2 + (B_t^y)^2)$. We claim that X_t is a local martingale. Define the function $f(x,y) = \log(x^2+y^2)$. Computing the partial derivatives we find

$$\partial_x f(x,y) = \frac{2x}{x^2+y^2}$$

$$\partial_{xx} f(x,y) = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\partial_y f(x,y) = \frac{2y}{x^2+y^2}$$

$$\partial_{yy} f(x,y) = \frac{2(x^2-y^2)}{(x^2+y^2)^2}$$

Then, we find by Ito Lemma that

$$dX_t = \frac{2B_t^x dB_t^x + 2B_t^y dB_t^y}{(B_t^x)^2 + (B_t^y)^2} + 0dt$$

holds, where the dt term is 0 since $\partial_{xx} f(x,y) + \partial_{yy} f(x,y) = 0$ holds. This formula shows that X_t is a local martingale. Then, by the Dubins-Schwartz Lemma from last class, we know that X_t is a time-change of BM. We claim that this means that planar Brownian motion is neighborhood recurrent but not recurrent.

Recall that 1-dimensional Brownian motion never reaches $\pm\infty$, yet it returns to any $a \in \mathbb{R}$ infinitely often over arbitrarily large times. This is a time-scale invariant property, meaning that no matter how we re-parametrize the process, this behavior will hold. Specifically, 1-dimensional Brownian motion will cross the interval between -1 and 1 infinitely many times without ever reaching ∞. Consequently, the process X_t must also cross between circles of radii r_1 and r_2 infinitely many times. Therefore, X_t cannot spend an infinite amount of time within a finite duration in the original parametrization, ensuring that it avoids both 0 and ∞.

$\square$

2.2 Conformal Invariance of Planar Brownian Motion

Recall that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if $\lim_{z\to 0} \frac{f(u+z)-f(u)}{z}$ exists $\forall u \in \mathbb{C}$. We also have that f is smooth as $f : \mathbb{R}^2 \to \mathbb{R}^2$.

If we take $z \to 0$ in $\mathbb{R}$ and $i\mathbb{R}$, we derive the Cauchy-Riemann Equations. Write $f(x+iy) = u(x,y)+iv(x,y)$, so we have that $u_x = v_y$ and $u_y = -v_x$. The usual elementary functions are holomorphic, so you can continue to do calculus in the way you'd hope.

Proposition 2.1. *Brownian motion is preserved by holomorphic functions up to a time change.*

Proof.

$$df(B_t^x + iB_t^y) = du(B_t^x, B_t^y) + i\,dv(B_t^x, B_t^y)$$

We need to understand each term separately.

$$du(B_t^x, B_t^y) = u_x dB_t^x + u_y dB_t^y + \frac{(u_{xx}+u_{yy})}{2}dt$$

$$dv(B_t^x, B_t^y) = v_x dB_t^x + v_y dB_t^y + \frac{(v_{xx}+v_{yy})}{2}dt$$

Now, let's apply the Cauchy Riemann Equations!
The dt terms cancel out since $u_{xx}+u_{yy} = v_{xy} - v_{xy} = 0$ and similarly, $v_{xx}+v_{yy} = -u_{xy}+u_{xy}$ we can ignore the drift part!

$$du(B_t, B_t^y) = u_x dB_t^x + u_y dB_t^y$$
$$dv(B_t, B_t^y) = v_x dB_t^x + v_y dB_t^y$$

This shows us that we have a local martingale. To show that these are a time change of Brownian Motion, we need to be able to say that $\mathrm{Corr}(du, dv) = 0$; i.e. we want to show that $u_x v_x + u_y v_y = 0$ holds. By Cauchy Riemann, we have $u_x v_x + u_y v_y = u_x v_x - v_x u_x = 0$. We also want to have the same amount of fluctuation; i.e. $u_x^2 + u_y^2 = v_x^2 + v_y^2$, which again follows by CR equations.

Thus, by a 2D version of Dubins-Schwartz, we have that

$$\begin{pmatrix} u(t)^2 & u(t)v(t) \\ u(t)v(t) & v(t)^2 \end{pmatrix} - A_t I_2$$

is a local martingale for

$$A_t = \int_0^t u_x(s)^2 + u_y(s)^2 ds = \int_0^t v_x(s)^2 + v_y(s)^2 ds$$

This means our quadratic variation component is just a scalar multiple of the identity. You can make this precise by using Ito's Lemma on $u(v)v(t)$ and see that $u(t)v(t)$ is a local martingale. Then, as before (when proving Dubins-Schwartz) we can define the stopping time τ_s such that $A_{\tau_s} = s$. Then we see that

$$s \mapsto (u(\tau_s), v(\tau_s))$$

is planar Brownian Motion by a 2-D version of Levy's characterization! □

Aside from being very cool, this also has neat consequences!

Application 1 Hitting Distribution for Planar Brownian Motion exit a disk or half-plane (this becomes considerably stronger when combined with the Riemann mapping theorem).
Clearly if $B_0 = 0$ the first exit location from the unit disk is uniform. Suppose $\vec{B}_0 = a \in \mathbb{R}$, for $a \in [0,1)$. Consider $f(\vec{B}_t)$ for the function $f(z) = \frac{z-a}{az-1}$, which is holomorphic on the unit disk.

Notice that $f(a) = 0$. This mapping also sends the boundary ∂D of the unit disk D to itself. To see this, we can note that $|z| = 1$ implies that $|f(z)| = 1$ and $|z| = 1 \iff \bar{z} = 1/2$.

Now, consider $f(B_t)$ which is Brownian Motion up to time change. Let τ be our exit time on the disk. Thus we have that $f(B_\tau) \sim \mathrm{Unif}(\partial D)$.

$$\mathrm{Law}(B_\tau) = f^{-1}(\mathrm{Unif}(\partial D))$$

which tells us exactly that

$$P_a(\theta) = \frac{1}{2\pi}\left(\frac{1-a^2}{1-2a\cos(\theta)+a^2}\right)$$

with θ denoting the angle. For those interested, $P_a(\theta)$ is a scaled Poisson kernel. What about starting a Brownian motion $\vec{B}_0 = i$ and asking for the law of the first real number we hit (i.e., the exit time from the upper half plane). Again, we can map between the disk and the upper half plane with a Mobius transformation. Consider

$$h(z) = i\left(\frac{1+z}{1-z}\right)$$

so $h(0) = i$ and if $|z| = 1$ then $h(z) \in \mathbb{R}$. In this case, the Law is given by $h(\mathrm{Unif}(\partial D))$ with corresponding density that is $\frac{1}{\pi(1+x^2)}$, which is the Cauchy density.

Interestingly, note that by starting from $\vec{B}_0 = 2i$ and tracking the first time that the imaginary part hits 1 and then 0, this shows that the sum of two Cauchy random variables is another Cauchy random variable (with different scaling).

Formally, Brownian motion is also conformal in higher dimensions, but you don't have as many nice conformal maps.

Application 2 Another proof of $\zeta(2) = \frac{\pi^2}{6}$. Consider the function $\phi(z) = \log(\frac{1+z}{1-z})$ which maps the unit disk D to a strip of length π. We know that the expected exit time of the strip is $\pi^2/4$ which we've seen before, from a 1-dimensional perspective. This tells us that

$$\mathbb{E}[|\phi(B_\tau)|^2] = \frac{\pi^2}{2}$$

We can write out a Taylor expansion of this function

$$\phi(z) = 2\left(z + \frac{z^3}{3} + \frac{z^5}{5} + \dots\right)$$

Then, for $U \sim \mathrm{Uniform}(\partial D)$ uniform on the complex unit circle, applying the two facts above, we have that

$$\mathbb{E}[|\phi(B_\tau)|^2] = \mathbb{E}[|\phi(U)|^2] = 4\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) = \frac{\pi^2}{2}.$$

This is because the Taylor expansion terms are orthonormal, i.e. $\mathbb{E}[\langle z^i, z^j\rangle] = 1_{i=j}$ for $i, j \geq 0$ where the inner product is taken in $\mathbb{R}^2$.

Dividing leads to

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Then, we notice that

$$\zeta(2) = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)\left(1 + \frac{1}{4} + \frac{1}{16} + \dots\right)$$

which follows from the fact that $\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2}$ and every positive integer m can be written uniquely as $m = 2^k \cdot$ some odd number. Then, applying the sum of a geometric series identity to the RHS, we find $1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{4}{3}$

Thus, we recover

$$\zeta(2) = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)\left(1 + \frac{1}{4} + \frac{1}{16} + \dots\right) = \frac{\pi^2}{2} \cdot \frac{4}{3} = \frac{\pi^2}{6}$$

which is precisely what we wanted to show.