Statistics 212: Lecture 4 (Feb. 5, 2025)

More on UI, L^p, Maximal Inequalities

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1 More on Uniform Integrability

We first recall an important result from last class. We have that the following conditions are equivalent for $S \subseteq L^1(\Omega, \mathscr{F}, P)$ to be uniformly integrable (U.I.).

Theorem 1.1. (Equivalent Conditions for U.I.)

(a) For all $\varepsilon > 0$, there exists $c > 0$ such that $\sup_{X \in S} \mathbb{E}[|X| \cdot 1_{|X| \geq c}] \leq \varepsilon$.

(b) For all $\varepsilon > 0$, there exists $c > 0$ such that $\sup_{X \in S} \mathbb{E}[(|X| - c)_+] \leq \varepsilon$.

(c) For all $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $A \in \mathscr{F}$ with $P(A) \leq \delta$, we have that $\sup_{X \in S} \mathbb{E}[|X| \cdot 1_A] \leq \varepsilon$.

We can now turn to prove another set of useful equivalent properties that were mentioned in the previous class but were not proven.

Theorem 1.2. Let $X_1, \ldots, X_n, \ldots$ be a sequence of L^1 random variables and $X_n \xrightarrow{\text{a.s.}} X$. Then the following are equivalent.

(a) $\{X_n : n \geq 1\}$ is U.I.

(b) $X \in L^1$ and $X_n \xrightarrow{L^1} X$.

(c) $X \in L^1$ and $E[|X_n|] \to E[|X|]$.

Proof. We first start with showing that (a) implies (b). Recall from last class that if $S \subseteq L^1(\Omega, \mathscr{F}, P)$ is U.I. then S is L^1-bounded; namely, $\sup_{X \in S} \mathbb{E}[|X|] < \infty$. By the U.I. assumption of (a), we have that $\sup_n \mathbb{E}[|X_n|] < \infty$ thus holds. Fix $\varepsilon > 0$. From here, we can note that by Fatou's Lemma

$$E[|X|] \leq \liminf_{n \to \infty} E[|X_n|] < \infty$$

holds. Certainly then $X \in L^1$. We now seek to show $X_n \xrightarrow{L^1} X$. To do so, for $c > 0$, define

$$\phi_c(u) = \min(c, \max(-c, u))$$

respectively. From here, we can let $X_n^c = \phi_c(X_n)$ and $X^c = \phi_c(X)$. Note that $X_n \xrightarrow{a.s.} X$ by assumption. By the continuous mapping theorem (CMT), it follows that $X_n^c \xrightarrow{a.s.} X^c$. Finally, by the dominated convergence theorem (DCT), it hence follows that

$$\lim_{n\to\infty} E[|X_n^c - X^c|] = 0$$

holds. From here, by the triangle inequality, we obtain

$$E[|X_n - X|] \le E[|X_n^c - X^c|] + E[|X - X^c|] + E[|X_n - X_n^c|]$$

We will now define $\varphi_c(u) = |u - \phi_c(u)| = (|u| - c)_+$. Fix $\varepsilon > 0$ arbitrary. Therefore, by the U.I. assumption, we have

$$E[|X_n - X_n^c|] = E[\varphi_c(X_n)] \le \varepsilon$$

for $c \ge c(\varepsilon)$ and for all n. Further, for c large enough, it follows by DCT that

$$E[|X - X^c|] \le \varepsilon$$

holds. By above, for $n \ge n(c, \varepsilon)$ large enough, we have

$$E[|X_n^c - X^c|] \le \varepsilon$$

To see why the above selection(s) of n and c are compatible, we note that we simply fix ε, choose c large enough depending on ε and then choose n large depending on n and ε. Combining all the terms above, we have that for $n \ge n(c, \varepsilon)$

$$E[|X_n - X|] \le 3\varepsilon$$

holds. So, we have shown $X \in L^1$ and $X_n \xrightarrow{L^1} X$, thereby showing that (a) implies (b).

We will now prove that (b) implies (c). Note that $|\cdot|$ is 1-Lipschitz. Hence, $||X_n| - |X|| \le |X_n - X|$ holds. By (b), $X_n \xrightarrow{L^1} X$. Thus, $\lim_{n\to\infty} E[|X_n - X|] = 0$, and so $\lim_{n\to\infty} E[||X_n| - |X||] = 0$. Finally, by the triangle inequality,

$$|E[|X_n|] - E[|X|]| \le E[||X_n| - |X||]$$

holds, thereby implying that $E[|X_n|] \to E[|X|]$, which is what we wanted to show.

Finally, we will show that (c) implies (a). Fix $\varepsilon > 0$. For any $c > 0$ and any n, we can consider the decomposition

$$E[|X_n|] = E[|X_n^c|] + E[(|X_n| - c)_+]$$

and

$$E[|X|] = E[|X^c|] + E[(|X| - c)_+]$$

respectively. By the overall assumption $X_n \xrightarrow{a.s.} X$, CMT and DCT, it follows that $E[|X_n^c|] \to E[|X^c|]$ as $n \to \infty$. By assumption of (c), we have $E[|X_n|] \to E[|X|]$. Thus, these two facts and the above decompositions imply that $E[(|X_n| - c)_+] \to E[(|X| - c)_+]$ holds. Now, we can note that, by DCT, for $c \ge c(\varepsilon)$ sufficiently large, we have that $E[(|X| - c)_+] \le \varepsilon$ holds. Thus taking $n \ge n(c, \varepsilon)$ sufficiently large, we obtain $E[(|X_n| - c)_+] \le 2\varepsilon$ since $E[(|X_n| - c)_+] \to E[(|X| - c)_+]$ holds as aforementioned.

We still need to handle the first finitely many X_k, but this is not a problem (for a similar reason that any finite collection of integrable random variables is UI). Namely for each $k < n(c, \varepsilon)$, we can choose $c_k > 0$ such that $E[(|X_k| - c_k)_+] \le \varepsilon$ holds. This is possible by DCT. Finally, take $c_* = \max(c(\varepsilon), c_1, \ldots, c_{n(c,\varepsilon)-1})$. By comparing with Theorem 1.1, we can see that $\{X_n : n \ge 1\}$ is U.I. □

1.1 Returning to Martingales

The above results were for general random variables (with a few additional conditions). We therefore now return to our treatment of martingales. To do so, we prove another theorem.

Theorem 1.3. Let $(X_n, \mathscr{F}_n)$ be an L^1 bounded martingale with $X_n \xrightarrow{a.s.} X_\infty$. Then the following are equivalent.

(a) $X_n \xrightarrow{L^1} X_\infty$.

(b) $\{X_n : n \geq 1\}$ is U.I.

(c) $E[|X_n|] \to E[|X_\infty|]$.

(d) $E[X_\infty | \mathscr{F}_n] = X_n$.

Proof. It is not too hard to see how the equivalence of (a), (b), and (c) can follow from Theorem 1.2. Hence, we will focus on (d).

We will first show that (a) implies (d). We know if $m \geq n$, then

$$E[X_m | \mathscr{F}_n] = X_n$$

holds by the definition of a martingale and the tower property. We know that condition expectation is L^1-contractive. Thus,

$$E[|E[X_\infty | \mathscr{F}_n] - E[X_m | \mathscr{F}_n]|] = E[|E[X_\infty - X_m | \mathscr{F}_n]|] \leq E[|X_\infty - X_m|]$$

holds. By assumption of (a), as $m \to \infty$, we have that $E[|X_\infty - X_m|] \to 0$ holds. So, by above

$$\lim_{m \to \infty} E[|E[X_\infty | \mathscr{F}_n] - E[X_m | \mathscr{F}_n]|] = 0$$

However, $E[X_\infty | \mathscr{F}_n]$ is constant with respect to m and so is $E[X_m | \mathscr{F}_n] = X_n$. So, in order for the above to hold, it must be the case that $E[X_\infty | \mathscr{F}_n] = X_n$.

We will now show that (d) implies (b). Recall that if $X \in L^1$ then $\{E[X|\mathscr{G}] : \mathscr{G} \subseteq \mathscr{F}\}$ is U.I. Note therefore

$$\{X_n : n \geq 1\} = \{E[X_\infty | \mathscr{F}_n] : n \geq 1\} \subseteq \{E[X_\infty | \mathscr{G}] : \mathscr{G} \subseteq \mathscr{F}\}$$

holds and so, if we can show that $X_\infty \in L^1$, then it follows by above that $\{X_n : n \geq 1\}$ is U.I, since it is a subset of a U.I. set. By Fatou's Lemma,

$$E[|X_\infty|] \leq \liminf_{n \to \infty} E[|X_n|] < \infty$$

holds, where we used the overall assumption that $\{X_n\}$ is L^1 bounded. Hence, $X_\infty \in L^1$ and we are done. □

1.2 Higher Moments

We now turn our attention towards higher moments. Firstly, we state the following Lemma, which we did not have time to prove in class, but we shall return to the proof in a future class.

Lemma 1.4. (L^p martingale maximal inequality) Fix $1 < p < \infty$ and let $(X_n)_{n \geq 1}$ be an L^p-bounded martingale. Denote $X^* = \sup_{n \geq 1} |X_n|$. Then, $X^* \in L^p$; in fact, $\|X^*\|_p \leq \frac{p}{p-1} \sup_n \|X_n\|_p$.

Theorem 1.5. Suppose $\{X_n : n \geq 1\}$ is L^p bounded for $1 < p < \infty$; i.e. $\sup_n \|X_n\|_p < \infty$, where $\|X_n\|_p = (E[|X_n|^p])^{1/p}$. Then, this implies the following:

(a) $\{X_n : n \geq 1\}$ is L^1 bounded.

(b) $\{X_n : n \geq 1\}$ is U.I.

(c) If $(X_n)_{\geq 1}$ is a martingale, then there exists a random variable X_∞ such that $X_n \xrightarrow{a.s.} X_\infty$ and $X_n \xrightarrow{L^p} X_\infty$.

Proof. To show (a), we can simply note that this follows from the argument that $\exists f(p) \in \mathbb{R}$ such that $|u| \leq |u|^p + f(p)$ for $p > 1$, and so the boundedness of the respective integrals follows.

We will now show (b). Fix $\varepsilon > 0$ arbitrary. By the definition of L^p-boundedness, $\exists M > 0$ such that $\|X_n\|_p \leq M, \forall n$. Note that

$$P(|X_n| \in [2^k, 2^{k+1}]) \leq P(|X_n|^p \geq 2^{kp}) \leq \frac{M^p}{2^{kp}}$$

holds by Markov's inequality and the definition of M. From here, we can observe that

$$\begin{aligned} E[|X_n| \cdot 1_{|X_n| \geq 2^j}] &= \sum_{k \geq j} E[|X_n| \cdot 1_{|X_n| \in [2^k, 2^{k+1})}] \\ &\leq \sum_{k \geq j} 2^{k+1-kp} \cdot M^p \\ &\leq M^p 2^{i+j-jp} \sum_{i \geq 0} 2^{i(1-p)} \\ &\leq C(p) \cdot 2^{j(1-p)} \\ &< \varepsilon \end{aligned}$$

holds, for j large enough and $C(p)$ is a constant depending on p. Hence, we are done with this implication also.

We now show (c). By the martingale convergence theorem, we have that $X_n \xrightarrow{a.s.} X_\infty$ for some X_∞. Hence, $|X_n - X_\infty|^p \xrightarrow{a.s.} 0$ also. Consider now X^* as defined in **Lemma 1.4**. By the corresponding Lemma and definition of X^*, we have that

$$|X_n - X_\infty|^p \leq |2X^*|^p$$

holds. Note that $|2X^*|^p \in L^1$ is integrable by the maximal inequality above, and crucially it does not depend on n. Hence, by DCT, we have that $\lim_{n\to\infty} E[|X_n - X_\infty|^p] = 0$, and so $X_n \xrightarrow{L^p} X_\infty$. □

With the above shown, we still require to prove **Lemma 1.4**. To do so, it turns out that we in fact need another Lemma.

Lemma 1.6. Let $(X_n)_{n\geq 1}$ be a submartingale. Then, for any $\lambda > 0$, we have that

$$\lambda \mathbb{P}(\max_{j \leq n} X_j \geq \lambda) \leq E[X_n \cdot 1_{\max_{j\leq n} X_j \geq \lambda}] \leq E[|X_n|]$$

holds.

Corollary 1.7. Let $(X_n)_{n\geq 1}$ be a non-negative submartingale. Then,

$$P(\max_{j \leq n} X_j \geq \lambda) \leq \frac{E[X_n]}{\lambda}$$

holds. We shall see in future classes that the above will help to prove **Lemma 1.4.**