
Statistics 212: Lecture 5 (February 10, 2025)

Doob's Inequality / L^p Maximal Inequality and Reverse Martingales

Instructor: Mark Sellke

Scribe: Théo Voldoire & Sarah McDonald

1 Doob's Inequality / L^p Maximal Inequality

In the previous lecture, we discussed the L^p Maximal Inequality. In this lecture, we will use Doob's Inequality to prove the L^p Maximal Inequality.

Theorem 1.1 (Doob's Inequality). *If (X_n) is a submartingale and $\lambda > 0$, then*

$$\lambda \mathbb{P}(\max_{j \leq n} X_j \geq \lambda) \leq \mathbb{E} \left[X_n \cdot \mathbf{1}_{\max_{j \leq n} X_j \geq \lambda} \right]$$

The proof of Doob's Inequality uses stopping times:

Proof. Let τ be first time t with $X_t \geq \lambda$, with $\tau = +\infty$ if that never happens. For each $0 \leq k \leq n$,

$$\mathbb{E}[X_\tau \cdot \mathbf{1}_{\tau \leq k}] \leq \mathbb{E}[X_n \cdot \mathbf{1}_{\tau \leq k}]$$

Now, summing over k ,

$$\mathbb{E}[X_\tau \cdot \mathbf{1}_{\tau \leq n}] \leq \mathbb{E}[X_n \cdot \mathbf{1}_{\tau \leq n}]$$

Note also that

$$\tau \leq n \Leftrightarrow \max_{j \leq n} X_j \geq \lambda$$

$$X_\tau \geq \lambda \text{ a.s. if } \tau \leq +\infty$$

Now, putting it all together, we have

$$\begin{aligned} \lambda \mathbb{P}(\max X_j \geq \lambda) &= \lambda \mathbb{P}(\tau \leq n) \\ &\leq \mathbb{E}[X_\tau \cdot \mathbf{1}_{\tau \leq n}] && \text{since } X_\tau \geq \lambda \\ &\leq \mathbb{E}[X_n \cdot \mathbf{1}_{\tau \leq n}] \\ &= \mathbb{E} \left[X_n \cdot \mathbf{1}_{\max_{j \leq n} X_j \geq \lambda} \right]. && \square \end{aligned}$$

Now, we will prove the L^p Maximum Inequality from last lecture (Lecture 4 Lemma 1.4) using Doob's Maximal Inequality.

Proof. First, let $U = U_n = \max_{j \leq n} |X_j|$, $V = V_n = |X_n|$. We will show $\|U_n\|_p \leq \frac{p}{p-1} \|V_n\|_p$.

Deducing L^p maximal inequality: first, $U_n \uparrow X^*$ as $n \rightarrow +\infty$ so $\|U_n\|_p \uparrow \|X^*\|_p$. Then, $\|V_n\|_p \uparrow \sup_{n \geq 1} \|X_n\|_p$ because $\mathbb{E}[|X_{n+1}|^p] \geq \mathbb{E}[|X_n|^p]$ as $u \mapsto |u|^p$ is convex.

Now, for fixed n ,

$$\begin{aligned} \|U\|_p^p &= \mathbb{E}[U^p] \stackrel{*}{=} p \int_0^\infty \lambda^{p-1} \mathbb{P}(U \geq \lambda) d\lambda \\ &\leq p \int_0^\infty \lambda^{p-2} \mathbb{E}[V \cdot 1_{U \geq \lambda}] d\lambda && \text{Doob's inequality, as } |X_n| \text{ is submartingale} \\ &\stackrel{\dagger}{=} \frac{p}{p-1} \int_{(V,U)} \left(\int_0^\infty \lambda^{p-2} V \cdot 1_{U \geq \lambda} d\lambda \right) d\mu \\ &= \frac{p}{p-1} \mathbb{E}[V U^{p-1}] \end{aligned}$$

where μ is the law of (V, U) . Step $*$ holds because for each fixed $u \geq 0$, we have $u^p = p \int_0^\infty \lambda^{p-1} 1_{u \geq \lambda} d\lambda = p \int_0^u \lambda^{p-1} d\lambda$. In other words, we are using Fubini to exchange the integration with respect to λ and the expectation with respect to U . Step \dagger is proved similarly, in reverse. (Note that the power of λ changed from $p-1$ to $p-2$ from Doob's inequality, which leads to the factor $\frac{1}{p-1}$ rather than $1/p$.)

Now introduce q s.t. $\frac{1}{p} + \frac{1}{q} = 1$, i.e. $q = \frac{p}{p-1}$, thus leading to

$$\begin{aligned} \frac{p}{p-1} \mathbb{E}[V U^{p-1}] &\leq \frac{p}{p-1} \|V\|_p \|U^{p-1}\|_q && \text{Holder's} \\ &= \frac{p}{p-1} \|V\|_p \|U\|_p^{p-1}. \end{aligned}$$

Canceling $\|U\|_p^{p-1}$ on both sides of the inequality, we obtain

$$\|U\|_p \leq \frac{p}{p-1} \|V\|_p,$$

as desired. □

Theorem 1.2 (Lévy's Upward Theorem). Fix $X \in L^1$, consider the filtration (\mathcal{F}_n) , and let $X_n = \mathbb{E}[X | \mathcal{F}_n]$. Then

$$X_n \xrightarrow{a.s., L^1} X_\infty = \mathbb{E}[X | \mathcal{F}_\infty],$$

where $\mathcal{F}_\infty = \sigma(\cup_n \mathcal{F}_n)$ is the smallest σ -algebra generated by (\mathcal{F}_n) .

Proof. X_∞ is \mathcal{F}_∞ -measurable as each X_n is \mathcal{F}_∞ -measurable. We want to show that for all $S \in \mathcal{F}_\infty$, the following property holds:

$$\mathbb{E}[X \cdot 1_S] = \mathbb{E}[X_\infty \cdot 1_S]. \quad (1)$$

First, property (1) holds if $S \in \mathcal{F}_n$,

$$\mathbb{E}[X \cdot 1_S] = \mathbb{E}[X_m \cdot 1_S] \quad \forall m \geq n,$$

as $X_m = \mathbb{E}[X | \mathcal{F}_m]$, $S \in \mathcal{F}_n \subseteq \mathcal{F}_m$. So, we find $\mathbb{E}[X \cdot 1_S] \rightarrow \mathbb{E}[X_\infty \cdot 1_S]$ as $m \rightarrow \infty$, since $X_m \xrightarrow{L^1} X_\infty$.

We need to use the π - λ -theorem. Let $\mathcal{G} = \{S \in \mathcal{F}_\infty : (1) \text{ holds for } S\}$. \mathcal{G} is a λ -system (closed under complement, countable disjoint union). Then, by what precedes, $\cup_n \mathcal{F}_n \subseteq \mathcal{G}$, which is a π -system. So by π - λ -Theorem, we find $\sigma(\cup_n \mathcal{F}_n) \subseteq \mathcal{G}$. □

A consequence of this result is the Kolmogorov 0-1 law.

Theorem 1.3 (Kolmogorov's 0-1 Law). *Let (X_n) be a sequence of iid random variables, $A \in \bigcap_n \sigma(X_n, X_{n+1}, \dots)$. Then $\mathbb{P}(A) \in \{0, 1\}$.*

Proof. Define $Y = 1_A$, $Y_n = \mathbb{P}(A | \sigma(X_1, \dots, X_n))$. A is $\sigma(X_1, \dots, X_n, \dots)$ -measurable, so

$$Y_n \longrightarrow \mathbb{P}(A | \sigma(X_1, \dots, X_n, \dots)) = 1_A$$

Thus, $Y_n = \mathbb{P}(A)$ for all n , as $A \in \sigma(X_{n+1}, X_{n+2}, \dots)$ is independent of $\sigma(X_1, \dots, X_n)$.

Hence, $\mathbb{P}(A) \xrightarrow{L^1} 1_A$. □

In the context of Brownian motion, we get the Blumenthal 0-1 law, so this construction is useful to understand hitting times, e.g. to show that hitting times are stopping times.

2 Reverse martingales

Definition 2.1 (Reverse Martingale). A reverse filtration is a sequence $\mathcal{F}_{-1} \supseteq \mathcal{F}_{-2} \supseteq \dots$ of decreasing filtrations. A reverse martingale is then a sequence of random variables X_{-1}, X_{-2}, \dots such that

- (a) $X_{-i} \in L^1$
- (b) X_{-i} is \mathcal{F}_{-i} measurable
- (c) $\mathbb{E}[X_{-i} | \mathcal{F}_{-i-1}] = X_{-i-1}$

Observation 2.2. *Any reverse martingale is uniformly integrable (UI), as*

$$X_{-i} = \mathbb{E}[X_{-i-1} | \mathcal{F}_{-i}].$$

Namely, recall that $\{\mathbb{E}[X | \mathcal{G}] : \mathcal{G} \subseteq \mathcal{F}\}$ is UI if $X \in L^1$. In this case, the time-reversal means every random variable in the sequence is a conditional expectation of X_{-1} .

Theorem 2.3 (Lévy's Downward Theorem). *Let $(X_{-i}, \mathcal{F}_{-i})$ be a reverse martingale. Then*

$$X_{-n} \longrightarrow X_{-\infty} = \mathbb{E}[X_{-1} | \mathcal{F}_{-\infty}], \text{ a.s. and in } L^1$$

with $\mathcal{F}_{-\infty} = \bigcap_n \mathcal{F}_{-n}$.

Proof. Almost sure convergence has the same proof as forward direction, using up-crossing inequalities. We then obtain L^1 convergence using UI.

Now, let's show that $X_{-\infty} = \mathbb{E}[X_{-1} | \mathcal{F}_{-\infty}]$.

Observe $X_{-\infty}$ is \mathcal{F}_{-n} -measurable $\forall n$ as X_{-n}, X_{-n-1}, \dots are \mathcal{F}_{-n} -measurable, thus $X_{-\infty}$ is $\mathcal{F}_{-\infty}$.

Next, we want that if $S \in \mathcal{F}_{-\infty}$, then $\mathbb{E}[X_{-\infty} \cdot 1_S] = \mathbb{E}[X_{-1} \cdot 1_S]$

We see that

$$\mathbb{E}[X_{-1} \cdot 1_S] = \mathbb{E}[X_n \cdot 1_S] \forall n,$$

as $S \in \mathcal{F}_{-n}$ and $X_{-n} = \mathbb{E}[X_{-1} | \mathcal{F}_{-n}]$.

Finally, we find $\mathbb{E}[X_{-n} \cdot 1_S] \rightarrow \mathbb{E}[X_{-1} \cdot 1_S]$ by L^1 convergence. □

We can prove the SLLN using reverse martingales.

Theorem 2.4 (Strong Law of Large Numbers). *Let X_1, X_2, \dots iid sequence of L^1 random variables. Then*

$$\bar{X}_n = \frac{S_n}{n} = \frac{X_1 + \dots + X_n}{n} \xrightarrow{\text{a.s., } L^1} \mathbb{E}[X_1].$$

Proof Sketch. Note that $\bar{X}_n = \mathbb{E}[X_1 | S_n, S_{n+1}, \dots]$. (This is a good exercise to check; idea is that $\mathbb{E}[X_1 | S_n, S_{n+1}, \dots] = \mathbb{E}[X_i | S_n, S_{n+1}, \dots]$ for each $i \leq n$ by symmetry.) Then (\bar{X}_n) is a reverse martingale (without the negative indexing), and so

$$\bar{X}_n \xrightarrow{\text{a.s., } L^1} \bar{X}_\infty = \mathbb{E} \left[X_1 \left| \bigcap_{n \geq 1} \sigma(S_n, S_{n+1}, \dots) \right. \right] \stackrel{?}{=} \mathbb{E}[X_1].$$

To show the last equality, use the Hewitt-Savage 0-1 Law (see Durrett) or Homework 2, Question 1. □