Statistics 212: Lecture 7 (Feb 19, 2025)

Construction of Brownian Motion

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1 Definition

Definition 1.1. A Brownian Motion on $t \in [0,1]$, is a random continuous function $B : [0,1] \to \mathbb{R}$ such that:

(a) $B_t - B_s \sim \mathcal{N}(0, t-s)$ if $t \geq s$

(b) if $t_1 \leq t_2 \leq \ldots \leq t_k$, then $(B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_k} - B_{t_{k-1}})$ are independent.

2 Questions Surrounding Definition

(a) **Existence:** Does such a function in Definition 1.1 exist as a $\mathscr{C}([0,1])$-valued random variable?

(b) **Uniqueness:** Is such a random function B unique?

(c) **Uncountable set:** How do we handle the uncountability of $[0,1]$?

To state these questions formally, we should have some probability measure μ on $\mathscr{C}([0,1])$ such that with $\varphi_t : \mathscr{C}([0,1]) \to \mathbb{R}$ given by $\varphi_t(f) = f(t)$, we should have $\text{Law}(\varphi_t(B) - \varphi_s(B)) \sim \mathcal{N}(0, t-s)$, etc, where $\varphi_t(B) = B_t$ and $\varphi_s(B) = B_s$.

Initial Attempt: One natural approach is to construct Brownian Motion from finite-dimensional distributions.

The followings are two thoughts that we may have when attempting to construct a Brownian Motion.

- Given $t_1, t_2, \ldots, t_n$, the defining property 2 of Definition 1.1 tells us the joint law of $\left(B_{t_1}, \ldots, B_{t_k}\right)$
- We should check if these distributions are consistent, i.e., if forgetting t_j, we can still recover correct law on $\left(B_{t_1}, \ldots, B_{t_{j-1}}, B_{t_{j+1}}, \ldots, B_{t_k}\right)$

Theorem 2.1 (Kolmogorov Extension (or Consistency) Theorem)**.** *There always exists a probability measure $\tilde{\mu}$ on $\mathbb{R}^{[0,1]}$ ($\equiv$ func$([0,1] \to \mathbb{R}$; i.e., the set of all functions from the interval $[0,1]$ to $\mathbb{R}$) which has all these finite-dimensional laws in the defining property 2 in Definition 1.1, given that these distributions are consistent.*

⇒ However, $\tilde{\mu}$ is not unique. For example, we can choose $u \sim \text{Unif}(0,1)$. We can start with $\tilde{\mu}$ but force $B_u = 100$. The stochastic process still obeys these defining properties 2 and the consistency property.

Problem: $\sigma-$algebra on $\mathbb{R}^{[0,1]}$ is generated by the evaluation mapping φ_t. In other words sets of the form $\{f \in \mathbb{R}^{[0,1]} \; f(t) \in (a,b)\}$ are measurable, and the σ-algebra is the one generated by these. Continuity of f is not even a measurable property.

3 Construction of Brownian Motion

3.1 Constructing a Sequence

To construct a Brownian Motion, we construct a sequence of piecewise linear interpolation. Specifically, we split the $[0,1]$ interval k times into $k+1$ equal intervals. For the trivial case where $k=0$, we have

$$B_t^0 = \begin{cases} 0 & t = 0, \\ z_0 & t = 1, \\ \text{linear interpolation} & \text{otherwise} \end{cases}$$

where $z_0 \sim \mathcal{N}(0,1)$.

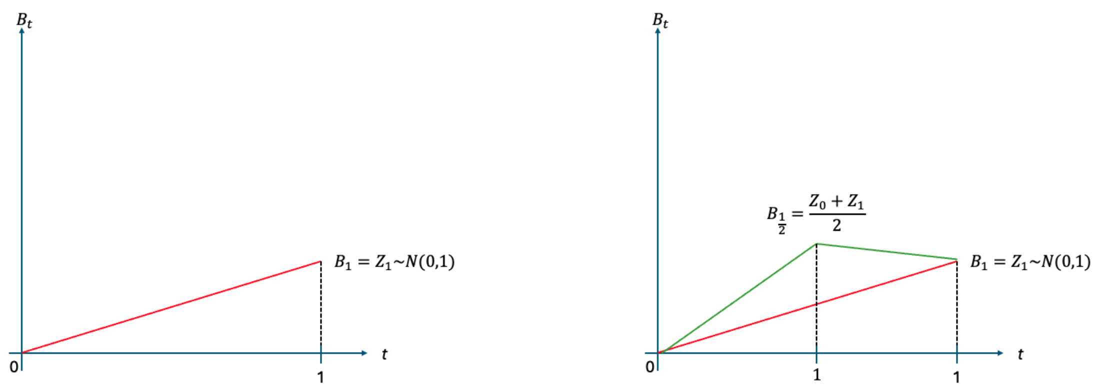

Figure 1: Case of $K=0$ (Left) and $K=1$ (Right)

Formally, we define B_t^k by

$$B_{j/2^k}^{k+1} = B_{j/2^k}^{k}, \quad \forall j \in \mathbb{Z}$$

$$\text{If } j \text{ is odd}:$$

$$B_{j/2^{k+1}}^{(k+1)} = \left(\frac{B_{(j-1)/2^{k+1}}^{(k)} + B_{(j+1)/2^{k+1}}^{(k)}}{2} \right) + \frac{Z_{k+1,j}}{\sqrt{2^{k+2}}}$$

where all Z_j's are IID. (Note that the first line exactly covers the j even case of the second line.)
We claim the following proposition:

Proposition 3.1. *Defining properties of Brownian Motion hold for $B^{(k)}$ at times $t_1, ..., t_i \in 2^{-k} \cdot \mathbb{Z}$*

Proof. The point is to induct on k. As Mark did in the class, we check the variance of new points, assuming things work so far (so we are doing a representative part of a full induction, some remaining parts are left to homework). That is,

$$\mathbb{E}\left[\left(B_{j/2^{k+1}}^{(k+1)} \right)^2 \right]$$

$$\mathbb{E}\left[\left(\frac{B^{(k)}_{(j-1)/2^{k+1}}+B^{(k)}_{(j+1)/2^{k+1}}}{2}\right)^2\right]+\mathbb{E}\left[\left(\frac{Z_{k+1,j}}{\sqrt{2^{k+2}}}\right)^2\right]$$

$$=\frac{1}{4}\left(\mathbb{E}\left[\underbrace{\left(B^{(k)}_{(j-1)/2^{k+1}}\right)^2}_{=\frac{j-1}{2^{k+1}}}+\underbrace{2B^{(k)}_{(j-1)/2^{k+1}}B^{(k)}_{(j+1)/2^{k+1}}}_{=\frac{2j-2}{2^{k+1}}}+\underbrace{\left(B^{(k)}_{(j+1)/2^{k+1}}\right)^2}_{=\frac{j+1}{2^{k+1}}}\right]\right)+\frac{1}{2^{k+2}}$$

and the last term is canceled out. □

Next, we will show $\{B^{(k)}\}$ is an almost surely Cauchy sequence with respect to d_{sup}. Hence, it has a limit B. To do this, we prove and use the following lemma:

Lemma 3.2. $\sum\limits_{k=0}^{\infty}\mathbb{E}[d_{sup}(B^k,B^{k+1})]<\infty$

Given this claim, we have $\forall\epsilon,\exists N(\epsilon,\omega)$, $\sum\limits_{k=N}^{\infty}d_{sup}(B^k,B^{k+1})\le\epsilon$. Consequently, we have $d_{sup}(B^M,B^L)\le\epsilon$, $\forall M,L\ge N$.
We also prove Lemma 3.2:

Proof of Lemma 3.2. Up to scale, it suffices to prove that $\mathbb{E}[\max\limits_{i=1}^{n}|Z_i|]\le O(\sqrt{\log(n)})$, where $\{Z_i\}$ are i.i.d. random variables following standard Gaussian distribution. Fix λ, and by Jensen's inequality,

$$e^{\lambda\mathbb{E}[\max_{i=1}^{n}|Z_i|]}\le\mathbb{E}[e^{\lambda\max|Z_i|}]\le\mathbb{E}[\sum_{i=1}^{n}e^{\lambda Z_i}+e^{-\lambda Z_i}]=2ne^{\lambda^2/2}$$
$$\Rightarrow\mathbb{E}[\max|Z_i|]\le\inf_{\lambda}\frac{1}{\lambda}\left(\frac{\lambda^2}{2}+\log(2n)\right).$$

Choosing $\lambda=\sqrt{\log(n)}$ gives the desired bound $\mathbb{E}[\max\limits_{i=1}^{n}|Z_i|]\le O(\sqrt{\log(n)})$.
Hence, $\mathbb{E}[d_{sup}(B^{(k)},B^{(k+1)})]=\frac{\max_j|Z_{k+1,j}|}{\sqrt{2^{k+2}}}\le O(\sqrt{k}\cdot 2^{-k/2})$. □

Remark. In fact, the limiting function B_t is $(\frac{1}{2}-\epsilon)$ Hölder $\forall\epsilon>0$, which means that $\sup_{t,s\in[0,1]}\frac{|B_t-B_s|}{|t-s|^{\frac{1}{2}-\epsilon}}<\infty\forall\epsilon>0$.

Proof. Here, we only give the outline of the overall proof. The direction is analogous to the previous one. Define $\|f\|_{C^{\frac{1}{2}-\epsilon}}=\sup_t|f(t)|+\sup_{t,s}\frac{|f(t)-f(s)|}{|t-s|^{\frac{1}{2}-\epsilon}}$, which is a complete metric space (but not separable). B^k is still Cauchy and is $\frac{\mathbb{E}[\max|Z_{k+1,j}|]}{2^{k/2}}\times 2^{k(\frac{1}{2}-\epsilon)}\approx\sqrt{k}2^{-k\epsilon}$, which is still summable. □

However, this metric space is not separable.

3.2 Desired Properties

Question (measurability): Why does this yield a probability measure on $C([0,1])$?

Proposition 3.3. *For each t, $B_t=\lim\limits_{k\to\infty}B_t^{(k)}$ is measurable with respect to the sequence of IID Gaussians $(Z_{k,j})$.*

Proof. B_t is an infinite weighted sum of $(Z_{k,j})$. □

Proposition 3.4. *Borel $\sigma-$algebra on $C([0,1])$ is exactly the one generated by evaluation functions $\varphi(t)$. In other words, the smallest σ-field on $C([0,1])$ such that all maps $\varphi(t)[B] = B_t$ are measurable is exactly the Borel $\sigma-$algebra.*

Specifically, letting F denote the "construction of Brownian motion" above (which results in a function $B = B_{[0,1]}$ from $[0,1] \to \mathbb{R}$) and ϕ_t the evaluation at time t, we have:

(a) $(Z_{k,j}) \xrightarrow{F} B \xrightarrow{\varphi_t} B_t \in \mathbb{R}$, *where $Z_{k,j}$ lies in probability space $(\Omega, \mathscr{F}, \nu)$.*

(b) $\varphi_t \circ F$ *is measurable $\forall\, t$ if and only if F is measurable wrt the Borel σ-algebra.*

(c) *As a consequence, letting ν be the product measure on our countably infinite family of Gaussians $Z_{k,j}$, the pushforward $\mu = F \circ \nu$ is well defined, and so we have constructed a genuine probability measure for Brownian motion on $C([0,1])$.*

and $A = \{S \subseteq C([0,1]) : F^{-1}(S) \in \mathscr{F}\}$ is a $\sigma-$field and $A \supseteq \varphi_t^{-1}((a,b)), \forall\, t, a, b. \Rightarrow A \supseteq \mathrm{Borel}(C([0,1]))$.

Proof. Each φ_t is continuous with respect to d_{sup}, hence it is measurable with respect to Borel $\sigma-$algebra $\Rightarrow \sigma(\varphi_t)_{t\in[0,1]} \subseteq \mathrm{Borel}(C([0,1])))$.

In the other direction, we claim that $\sigma(\varphi_t)_{t\in[0,1]}$ contains open balls $\{f : d_{sup}(f,g) < \epsilon\} = B_\epsilon(g)$. Indeed, we can write

$$B_\epsilon(g) = \bigcup_{n\geq 1} \bigcap_{q\in\mathbb{Q}} \left\{ f : |f(q) - g(q)| < \epsilon - \frac{1}{n} \right\}.$$

(Here the $1/n$ terms are needed in case e.g. $|f(x) - g(x)| = \epsilon$ holds at exactly one value of x which is irrational.) □