Statistics 212: Lecture 9 (Feb, 2025)

Continuous Time Stochastic Processes

Instructor: Mark Sellke

Scribe: Gianfranco Randazzo and Mohammad Khan

So far we have defined BM as a random function in $C([0,1])$ or $C([0,\infty))$. Today we will look at BM as a Markov Chain in continuous time. Now a question is how to justify optional stopping in continuous time?

1 Continuous Time Markov Chains

Definition 1.1. $\{X_n, n \geq 1\}$ is Markov Chain if:

$$X_n \perp\!\!\!\perp (X_1, \dots, X_{n-2} \mid X_{n-1}) \iff \mathbb{P}(X_n \in A \mid X_1, \dots, X_{n-1}) = \mathbb{P}(X_n \in A \mid X_{n-1})$$

Definition 1.2. A continuous time filtration is a family of σ-algebras, $\{\mathscr{F}_t, t \in \mathbb{R}_{\geq 0}\}$, $\mathscr{F}_s \subset \mathscr{F}_t$ if $s \leq t$. An example is $\mathscr{F}_t = \sigma(\{B_s : 0 \leq s \leq t\}) = \sigma(B_{[0,t]})$.

Claim. BM satisfies Markov Chain in continuous time.

Note that for any $s \geq 0$, $X_t = B_{t+s} - B_s$ is BM independent of past points.

- Idea 1. Construct $B_{(0,\infty)}$ as a continuation of $B_{[0,s]}$ and $X_{(s,\infty)}$, see that you get BM
- Idea 2. Check finite dimensional distributions, more $\pi - \lambda$.

Definition 1.3. Call $X \in C([0,\infty))$ *adapted* to filtration $\mathscr{F}$ if X_t is $\mathscr{F}_t$-measurable $\forall t$.

This implies that $X_{(0,t]} \in C([0,t])$ is $\mathscr{F}_t$-measurable since finite dimensional projections are $\mathscr{F}_t$-measurable.

Definition 1.4. We say $\mathscr{F}^+$ is right continuous if $\mathscr{F}_t^+ = \bigcap_{s>t} \mathscr{F}_s, \forall t$.

This is generally benign. If $\mathscr{F}$ is a filtration, then we have $\mathscr{F}_t^+ = \bigcap_{s>t} \mathscr{F}_s$ gives a right continuous filtration.

Theorem 1.5 (Blumenthal 0-1 Law)**.** *For BM if $A \in \mathscr{F}_t^+$, then $\mathbb{P}(A \mid \mathscr{F}_t) \in \{0,1\}$-valued $\mathscr{F}_t$-measurable function. Equivalently, changing from $\mathscr{F}_t \to \mathscr{F}_t^+$ only adding measure $\{0,1\}$ sets.*

Explicit deduction from Kolmogorov 0-1:

$$\begin{aligned}
B_{[t+1/2,t+1]} &\leftarrow X_s^{(1)} = B_{t+1/2+s} - B_{t+1/2}, \quad s \in [0,1/2] \\
B_{[t+1/4,t+1/2]} &\leftarrow X_s^{(2)} = B_{t+1/4+s} - B_{t+1/4}, \quad s \in [0,1/4] \\
B_{[t+1/8,t+1/4]} &\leftarrow X_s^{(3)} = B_{t+1/8+s} - B_{t+1/8}, \quad s \in [0,1/8] \\
&\vdots
\end{aligned}$$

Note that $A \in \sigma(B_{[0,t]}, X_s^{(1)}, X_s^{(2)}, \dots)$. Then we have that A is independent of the first k $X_s^{*(i)}, \forall k$.

1.0.1 Applications w.p. 1

(a) $\{\forall \epsilon > 0, \max_{0\le s\le\epsilon} B_s > 0\} \in \mathscr{F}_0^+$, and occurs w.p. 1

(b) $\{\forall \epsilon > 0, \exists \delta \in (0,\epsilon)$ such that $B_s = 0\} \in \mathscr{F}_0^+$, and occurs w.p. 1

(c) BM has a local max in $(0,\epsilon] \implies$ BM has a dense set of local max, and occurs w.p. 1

Proof. Letting $\mathscr{F}_0 = \{\varnothing, \Omega\}$. For 1, assume that BM is not constant on any interval $[0,\epsilon]$ (we saw non-differentiable). Then either:

$$D_\epsilon = \left\{\max_{0\le s\le\epsilon} B_s > 0\right\} \text{ or } E_\epsilon = \left\{\max_{0\le s\le\epsilon} -B_s > 0\right\}$$

holds $\forall\epsilon$. If $\delta < \epsilon$, then E_ϵ false $\implies$ E_δ false. Similarly, for D_δ, D_ϵ. This if $E_\epsilon, D_{\epsilon_2}$ both false, then $\delta = \min(\epsilon_1, \epsilon_2)$, then E_δ, D_δ both false implies that BM continuous on $[0,\delta]$. Thus w.p. 1 either $\{E_\epsilon, \forall\epsilon\}$ or $\{D_\epsilon, \forall\epsilon\}$.

For 2, let $A_\epsilon = \{\exists\delta \in (0,\epsilon) : B_\delta = 0\}$. Now we will let $A = \bigcap_{\epsilon>0} A_\epsilon \in \mathscr{F}_0^+$. We note this probability is not a function of ϵ because we can rescale our path, i.e. $\mathbb{P}(A_\epsilon) = p$ which doesn't depend on ϵ. Then we have that $\mathbb{P}(A) = \lim_{\epsilon\downarrow 0} \mathbb{P}(A_\epsilon) = p = \{0,1\}$, complements from MCT. Now we want to justify that $p \ne 0$. We note that $\mathbb{P}(A_\epsilon) > 0$ as $\{B_{\epsilon_2} > 0, B_\epsilon < 0\}$, has positive probability, IVT implies existence of a 0. □

1.1 Stopping Time

Definition 1.6. A non-negative random variable T (on the same probability space as BM) is a stopping time for $(B_t, \mathscr{F}_t^+)$ if $\{T \le t\} \in \mathscr{F}_t, \forall t$.

Claim. If $\mathscr{F}_t^+$ right-continuous, then equivalently to require $\{T < t\} \in \mathscr{F}_t^+$

Proof. Assume $\{T < t\} \in \mathscr{F}_t^+ \implies \{T \le t\} = \bigcap_{n\ge 1/\epsilon}\{T < t + 1/n\} \in \bigcap_{s>t}\mathscr{F}_s^+ = \mathscr{F}_{t+\epsilon}^+, \forall\epsilon$. Check the other direction. □

Theorem 1.7 (Strong Markov Property)**.** *It says, informally, the future after stopping time is a fresh BM. Thus, let T be a stopping time for BM. Then $W_t = B_{T+t} - B_T$ is a BM, also independent of $\mathscr{F}_T^+ = \{A \in \sigma(\{B_s, s \ge 0\}) : A \cap \{T \le t\} \in \mathscr{F}_t^+, \forall t\}$. Note that T is random.*

Also, given T a stopping time, then $T_t = \min(T, t)$ is also a stopping time for any fixed t, and $\{\mathscr{F}_{T_t} : t \ge 0\}$ is called the "stopped filtration".

Proof. First step is to assume $T \in \{t_1 \le t_2 \le t_3 \le \dots\}$ a.s. takes values in a countable set. Then we can consider each t_j separately. Then we have that $\mathbb{P}(T = t_j) = p_j > 0$. We want: conditionally on $\{T = t_j\}$, future is a BM. This follows from the ordinary Markov Property at time t_j. Discreteness $\implies \{T = t_j\} \in \mathscr{F}_{t_j}^+$. This says that $W_s^{(i)} = B_{t_j+s} - B_{t_j} \perp\!\!\!\perp \mathscr{F}_{t_j}$ also $W_s^{(i)} \perp\!\!\!\perp \mathscr{F}_{t_j}$. Full proof next time will be to approximate any stopping time by a discrete one like this, and use continuity to take the limit. □