Critical Points of Gaussian Fields

Finiteness of Moments

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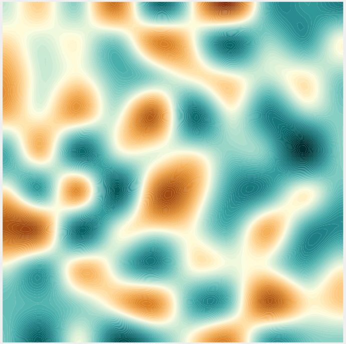

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Introduction

Let $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ be a random field.

Example (Bargmann-Fock random field).

Let $(\gamma_\alpha)_{\alpha\in\mathbb{N}^d}$ be a family of i.i.d. Gaussian variates. Then,

$$\varphi\colon \mathbb{R}^d \longrightarrow \mathbb{R}, \quad \varphi(x) = e^{-\|x\|^2/2} \sum_{\alpha\in\mathbb{N}^d} \frac{\gamma_\alpha}{\sqrt{\alpha!}} x^\alpha$$

is a *smooth Gaussian* random field, known as the *Bargmann-Fock random field.*

Let $B = B(0,R) \subset \mathbb{R}^d$ be the closed ball of radius R and center 0. The variate of interest is

$$X = X(f,R) = \#\{x \in B \mid \nabla f(x) = 0\},$$

the *number of critical points* of f contained in B.

Moment Conjecture

So far, the exact distribution of X is out of reach, and much research is instead focused on understanding its *moments*.

Conjecture.

Assume that the covariance function of the Gaussian random field $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, as well as all of its derivatives, are square-integrable. Then, for all p,

$$\lim_{R\to\infty} \mathbb{E}\left[\left(\frac{X - \mathbb{E}[X]}{\sqrt{\mathbb{V}[X]}}\right)^p\right] = \mathbb{E}[\gamma^p], \quad \gamma \sim \mathcal{N}(0,1).$$

In the special case $d = 1$:

- Conditions for finite moments have been precisely established by Cuzick (1975) as well as Armentano et al. (2020).
- Results on moment asymptotics include Nazarov and Sodin (2015) and Gass (2023).

Differentials and Jets

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, define the *differential operator*

$$\partial^\alpha : C^{|\alpha|}(\mathbb{R}^d) \to C^{|\alpha|}(\mathbb{R}^d), \quad \partial^\alpha f(x) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f(x_1, \ldots, x_d), \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.$$

Then, $(\partial^\alpha f(x))_{|\alpha| \leq p}$ is the *p-jet of f at x*. It is a vector of length

$$\sum_{i=0}^{p} \binom{d+i-1}{i}.$$

Example (The case $d = 2$ and $p = 2$).

The 2-jet of $f : \mathbb{R}^2 \to \mathbb{R}$ at (x_1, x_2) is

$$\left(f(x_1, x_2), \frac{\partial}{\partial x_1} f(x_1, x_2), \frac{\partial}{\partial x_2} f(x_1, x_2), \frac{\partial^2}{\partial x_1^2} f(x_1, x_2), \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} f(x_1, x_2), \frac{\partial^2}{\partial x_2^2} f(x_1, x_2) \right).$$

Main Result

Theorem (Gass and Stecconi, 2023).

Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a C^{p+1} Gaussian random field, and assume that

$$\det \mathrm{Cov}((\partial^\alpha f(x))_{|\alpha| \le p}) > 0$$

for all $x \in B$, i.e. that the *p-jets of f are nondegenerate.* Then, $\mathbb{E}[X^p] < \infty$.

It follows from Azais and Wschebor (2009) that the Bargmann-Fock random field $\varphi\colon \mathbb{R}^d \to \mathbb{R}$ from before has nondegenerate p-jets for all p, so the moments of $X(\varphi, R)$ are all finite.

Previously, Beliaev, McAuley, and Muirhead (2022) established the special case $p = 3$ using a technical divided difference method, and no results were known for moments of order $p \geq 4$ in dimension $d \geq 2$.

Proof Reductions

- By the compactness of B, it suffices to establish the result when B is an arbitrarily small compact neighborhood of x, for all x. In other words, *the statement is local.*

- The result is true for any Gaussian random polynomial $g\colon \mathbb{R}^d \to \mathbb{R}$ by Bezout's theorem (Bochnak, Coste, and Roy, 2013).

- There is a universal constant $C_p > 0$ such that

$$\mathbb{E}[X^p] \leq C_p(1 + \mathbb{E}[X^{[p]}]),$$

  where $X^{[p]} = X(X-1)\cdots(X-p+1)$ is the *p-factorial power of* X. Therefore, it suffices to prove that $\mathbb{E}[X^{[p]}]$ is finite.

Kac-Rice Formula

Let $\Delta = \{x = (x_1, \ldots, x_p) \in (\mathbb{R}^d)^p \mid \exists i \neq j \text{ s.t. } x_i = x_j\}$ denote the *fat diagonal* (in $(\mathbb{R}^d)^p$). The following version of the Kac-Rice formula can be found in Azais and Wschebor (2009).

Theorem (Kac-Rice formula).

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^2 Gaussian random field such that $(\nabla f(x_i))_{1 \leq i \leq p}$ is nondegenerate for all $x = (x_1, \ldots, x_p) \in B^p - \Delta$, say with density $\psi_{f,x}$. Then,

$$\mathbb{E}[X^{[p]}] = \int_{B^p - \Delta} \rho_f(x)\, dx,$$

where $\rho_f(x)$ is

$$\mathbb{E}\left[\prod_{k=1}^{p} |\det \nabla^2 f(x_k)| \;\middle|\; \nabla f(x_1) = \cdots = \nabla f(x_p) = 0\right] \psi_{f,x}(0).$$

Density Decomposition

Gathering the proof reductions from before, it suffices to prove the following.

Lemma.

For sufficiently small R and all $x \in B^p - \Delta$,

$$\rho_f(x) = Q(x)\sigma_f(x),$$

where Q is *universal* (meaning it does not depend on f) and σ_f is bounded above and below by *positive* constants.

Once this is established,

$$\rho_f \leq \frac{\sup \sigma_f}{\inf \sigma_g} \rho_g \in L^1$$

for any nondegenerate Gaussian random polynomial $g\colon \mathbb{R}^d \to \mathbb{R}$.

Main Obstruction

The main obstruction is that it is difficult to understand the behavior of ρ_f near Δ. Namely,

$$\rho_f(x) \propto \frac{\mathbb{E}\left[\prod_{k=1}^{p} |\det \nabla^2 f(x_k)| \;\middle|\; \nabla f(x_1) = \cdots = \nabla f(x_p) = 0\right]}{\sqrt{\det \operatorname{Cov}(\nabla f(x_1), \ldots, \nabla f(x_p))}},$$

and the challenge is understanding the near-diagonal degeneracy of the vectors

$$(\nabla f(x_1), \ldots, \nabla f(x_p), \nabla^2 f(x_k)), \quad 1 \leq k \leq p.$$

When $d = 1$, this can be tackled with a divided differences trick as well as Hermite-Lagrange interpolation (Gass, 2023; Armentano et al., 2020; Ancona and Letendre, 2021).

Yet, in higher dimensions, there is no *well-poised* interpolation, meaning no *unique* polynomial of degree $p - 1$ interpolating a function at p unique points (Davis, 1975). The key insight from Gass and Stecconi (2023) is that divided differences is *secretly* a Gram-Schmidt process.

Gram-Schmidt Process

Example (Gram-Schmidt process for $d = 1$).

Let δ_x be the evaluation map at x. For $x = (x_1, \ldots, x_p) \in \mathbb{R}^p - \Delta$,

$$\delta_x = \begin{pmatrix} \delta_{x_1} \\ \vdots \\ \delta_{x_p} \end{pmatrix} = A(x) \begin{pmatrix} \frac{\delta_{x_1}}{\|\delta_{x_1}\|} \\ \vdots \\ \frac{\delta_{x_p} - \mathrm{Proj}_{\mathrm{Span}(\delta_{x_1},\ldots,\delta_{x_{p-1}})}(\delta_{x_p})}{\|\delta_{x_p} - \mathrm{Proj}_{\mathrm{Span}(\delta_{x_1},\ldots,\delta_{x_{p-1}})}(\delta_{x_p})\|} \end{pmatrix}.$$

Evaluating at a function f yields

$$\delta_x f = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_p) \end{pmatrix} = A(x) \begin{pmatrix} f[x_1] \\ \vdots \\ f[x_1, \ldots, x_p] \end{pmatrix}.$$

Example (Gram-Schmidt process for general d).

For $x = (x_1, \dots, x_p) \in (\mathbb{R}^d)^p - \Delta$,

$$\delta_x \nabla f = \begin{pmatrix} \nabla f(x_1) \\ \vdots \\ \nabla f(x_p) \end{pmatrix} = Q_0(x) N_f(x),$$

where:

- $Q_0(x)$ is a *universal* square matrix of size dp.
- $N_f(x)$ is a vector of dp *orthonormal* linear forms depending on f.

Then, by properties of the determinant,

$$\sqrt{\det \mathrm{Cov}(\nabla f(x_1), \dots, \nabla f(x_p))} = |\det Q_0(x)| \sqrt{\det \mathrm{Cov}(N_f(x))}.$$

Decomposition Achieved

Moreover, one can show that for suitable $H_{f,k}$ and *universal* Q_k $(1 \leq k \leq p)$,

$$\mathbb{E}\left[\prod_{k=1}^{p} |\det \nabla^2 f(x_k)| \;\middle|\; \delta_x \nabla f = 0\right] = \left(\prod_{k=1}^{p} Q_k(x)\right) \mathbb{E}\left[\prod_{k=1}^{p} |H_{f,k}(x)| \;\middle|\; N_f(x) = 0\right].$$

Therefore,

$$\rho_f(x) \propto \underbrace{\frac{\prod_{k=1}^{p} Q_k(x)}{|\det Q_0(x)|}}_{Q(x)} \underbrace{\frac{\mathbb{E}\left[\prod_{k=1}^{p} |H_{f,k}(x)| \;\middle|\; N_f(x) = 0\right]}{\sqrt{\det \mathrm{Cov}(N_f(x))}}}_{\sigma_f(x)},$$

and the desired decomposition of the Kac-Rice density is achieved.

Remark.

The existence of an adequate scalar product for evaluation maps was implicitly assumed in the analysis thus far. This can be justified with the introduction of *Kergin interpolation*.

Kergin Interpolation

Let $\mathbb{R}_p[X_1, \ldots, X_d]$ be the space of real polynomials of degree *at most p* in $X_1, \ldots, X_d$.

Theorem (Kergin, 1980).

For $x = (x_0, x_1, \ldots, x_p) \in (\mathbb{R}^d)^{p+1}$, there is a projector

$$\Pi_x \colon C^p(\mathbb{R}^d) \longrightarrow \mathbb{R}_p[X_1, \ldots, X_d]$$

such that if the multiplicity of x_k in x is n, then $\partial^\alpha(\Pi_x f)(x_k) = \partial^\alpha f(x_k)$ for all $|\alpha| < n$.

Thus, $\delta_x \nabla$ can be viewed as a family of linear forms on a finite-dimensional vector space and there exists a scalar product (and corresponding *norm* $\|\cdot\|$) on this space.

Meanwhile, the boundedness of σ_f follows from the nondegeneracy of the p-jets of f plus a technical argument using the Bolzano-Weierstrass theorem.

Alternative Approach

This result was also proven (independently, around the same time, and using an alternative approach) by Ancona and Letendre (2023). Some aspects of their proof are given below.

- The space $(\mathbb{R}^d)^p - \Delta$ can be completed to a *compact* space $C_p[\mathbb{R}^d]$ so that if $f : \mathbb{R}^d \to \mathbb{R}$ is a Gaussian random field satisfying the theorem assumptions, then

  $$F\colon (\mathbb{R}^d)^p - \Delta \longrightarrow (\mathbb{R}^d)^p, \quad F(x_1, \ldots, x_p) = (\nabla f(x_1), \ldots, \nabla f(x_p))$$

  can be extended to a Gaussian random field

  $$F^{\times}\colon C_p[\mathbb{R}^d] \to (\mathbb{R}^d)^p$$

  having the same zeros as F.

- This compactification is obtained by a sequence of *blow-ups* using Hironaka's theorem on the *resolution of singularities.*

Result for Zeros

For a random field $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$ now, let

$$Z = Z(f, R) = \#\{x \in B \mid f(x) = 0\}$$

be the *number of zeros* of f contained in B.

Theorem (Gass and Stecconi, 2023).

Let $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a C^p Gaussian random field, and assume that

$$\det \mathrm{Cov}((\partial^\alpha f(x))_{|\alpha| \le p-1}) > 0$$

for all $x \in B$. Then, $\mathbb{E}[Z^p] < \infty$.

Note this *does not* imply the previous theorem because second derivative symmetry prevents the vector $(\partial^\alpha \nabla f(x))_{|\alpha| \le p-1}$ from being nondegenerate.

More Extensions

Gass and Stecconi (2023) and Ancona and Letendre (2023) also exhibit analogous results on the finiteness of moments of Z (resp. X) when:

- f is a holomorphic Gaussian random field on $\mathbb{C}^d$.
- f is a C^p (resp. C^{p+1}) Gaussian random field on a smooth Riemannian manifold.
- f is a holomorphic Gaussian random field on a complex Riemannian manifold.

This suggests that deeper underlying ideas are at play. To bring these themes to the fore, Gass and Stecconi (2023) introduce *p-interpolating spaces* and prove a general result that contains all the results seen so far as *special cases.*

Let $\delta_x \colon C^0(\mathbb{R}^d, \mathbb{R}^d) \to (\mathbb{R}^d)^p$ be the evaluation map $\delta_x f = (f(x_1), \ldots, f(x_p))$ from above. This can be viewed as dp linear forms on $C^0(\mathbb{R}^d, \mathbb{R}^d)$.

Let $\mathcal{J}_x \colon C^1(\mathbb{R}^d, \mathbb{R}^d) \to \mathbb{R}$ be the map $\mathcal{J}_x f = \det \nabla f(x)$. This can be viewed as a polynomial of degree d on $C^1(\mathbb{R}^d, \mathbb{R}^d)$.

Interpolating Space

Definition (p-interpolating space).

A finite-dimensional subspace $V \subset C^1(\mathbb{R}^d, \mathbb{R}^d)$ is called a *p-interpolating space* if for all points $y = (y_1, \ldots, y_p) \in (\mathbb{R}^d)^p - \Delta$:

- A There is a subspace $V_0 \subset V$ such that $\delta_y(V_0) = (\mathbb{R}^d)^p$.
- B The polynomials $(\mathcal{J}_{y_k})_{1 \le k \le p}$ are nonzero on $\mathrm{Ker}(\delta_y) \cap V$.
- C For every closed ball $B \subset \mathbb{R}^d$, there is a constant C_B and a subset $N_B \subset V$ such that for all $g \in V - N_B$,

$$\#\{x \in B \mid g(x) = 0\} \le C_B.$$

Let g be a nondegenerate Gaussian vector with values in V.

- A ensures that one can write the Kac-Rice formula for g.
- B ensures that the Kac-Rice density for g never vanishes.
- C endows g with the behavior of a Gaussian random polynomial.

Adaptedness and Strength

Definition (Adapted p-interpolating space).

The space V is a *p-interpolating space adapted to a subspace* $W \subset C^p(\mathbb{R}^d, \mathbb{R}^d)$ if:

1. V is a p-interpolating space.
2. For all $x = (x_1, \dots, x_p) \in (\mathbb{R}^d)^p$, there is a continuous linear map $\mathcal{K}_x^k \colon W \longrightarrow V$ such that $\mathcal{K}_x^0(W) = V_0$ and for all $f \in W$,

$$\delta_x f = \delta_x \mathcal{K}_x^k f \text{ and } \mathcal{J}_{x_k} f = \mathcal{J}_{x_k} \mathcal{K}_x^k f.$$

3. For all $f \in W$, the map $x \mapsto \mathcal{K}_x^k f$ is continuous.

Call the family $\mathcal{K} = (\mathcal{K}_x^k)_{x \in (\mathbb{R}^d)^p, 1 \leq k \leq p}$ a *p-interpolator between W and V*.

Call $\mathcal{K}$ a *strong* p-interpolator if the $\mathcal{K}_x^k$ are all surjective.

Call V a *strong* p-interpolating space if there is a strong p-interpolator between V and itself.

General Result

Theorem (Gass and Stecconi, 2023).

Let $f\colon \mathbb{R}^d \to \mathbb{R}^d$ be a C^p Gaussian random field and W the support of the Gaussian measure on $C^p(\mathbb{R}^d, \mathbb{R}^d)$ associated to f. Let V be a strong p-interpolating space adapted to W. Then, the Kac-Rice formula for f holds, and there exists a C^p function

$$Q = Q_V \colon (\mathbb{R}^d)^p - \Delta \longrightarrow \mathbb{R}_+$$

depending only on V and satisfying the following properties:

- For any closed ball $B \subset \mathbb{R}^d$, Q is integrable on $B^p - \Delta$.
- There is a positive constant $C_f > 0$ such that $\rho_f \leq C_f Q$.
- If the p-interpolator between W and V is strong, then there is a positive constant $c_f > 0$ such that $c_f Q \leq \rho_f$.

In particular, $\mathbb{E}[Z^{[p]}] < \infty$ for every closed ball $B \subset \mathbb{R}^d$.

Thank You

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