# Statistics 291: Lecture 11 (February 27, 2024) Bounds on $F_N(\beta)$ in the High-Temperature Case with External Field

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### 1 Introduction

Today we will introduce a new technique for studying models on the sphere: restricting  $H_N$  on subspherical bands. In particular, if one considers a band in  $S_N$ , the restriction of  $H_N$  will also be a spin glass, in one lower dimension.

In this class, we will concern ourselves with the high temperature case, with an external field applied. In particular for any non-zero external field, this includes the restriction of  $H_N$  to subspheres  $qS_N$  for q small enough. Next time we will cover shattering for pure p-spin models.

#### 2 Recap

Consider the general *p*-spin model

$$H_N(x) = \sum_{p=1}^P \gamma_p H_{N,p}(x),$$

where

$$\mathbb{E}H_N(x)H_N(y) = N\xi(R(x,y)), \qquad \xi(R) = \sum_{p=1}^P \gamma_p^2 R^p.$$

In the case of high-temperature with no external field, we have  $\gamma_1 = 0$  and

$$\frac{1}{N}\log\mathbb{E}Z_N(\beta) = \beta^2\xi(1)/2, \qquad \frac{1}{N}\log\mathbb{E}Z_N(\beta)^2 = \max_{-1 < R < 1}\beta^2(\xi(1) + \xi(R)) + \frac{1}{2}\log(1 - R^2).$$

Setting  $\Phi(R) = \beta^2(\xi(1) + \xi(R)) + \frac{1}{2}\log(1 - R^2)$ , we can draw a rough sketch to see that the maximum of  $\Phi(R)$  is attained at R = 0. This relies on  $\gamma_1 = 0 \iff \xi'(0) = 0$  and the assumption  $\beta \le \beta_0(\xi)$ . Thus

$$\frac{1}{N}\log \mathbb{E} Z_N(\beta)^2 \approx \beta^2 \xi(1).$$

We've computed the first and second moments of the partition function, the first moment exactly and the second moment to leading exponential order. By Paley-Zygmund and the Concentration of  $F_N(\beta)$ , we also obtained the bounds

$$\beta^2 \xi(1)/2 - o(1) \le \mathbb{E}F_N(\beta) = \frac{1}{N} \mathbb{E}\log Z_N(\beta) \le \frac{1}{N} \log \mathbb{E}Z_N(\beta) = \beta^2 \xi(1)/2.$$

However, if  $\gamma_1 > 0$ , then the maximum of  $\Phi(R)$  will be achieved at some positive *R*, thus making it harder to find the second moment of  $Z_N(\beta)$ .

Another argument that the annealed free energy ought to be incorrect is as follows. Recall from lecture 4 that Gaussian integration by parts gives:

$$\frac{d}{d\beta}F_N(\beta) = \beta \cdot \left(\xi(1) - \mathbb{E}_{x,x' \sim \mu_\beta}[R(x,x')]\right).$$

This suggests that

$$F_N(\beta) \approx \beta^2 \xi(1)/2 \iff \mathbb{E}_{x,x' \sim \mu_\beta}[\xi(R(x,x'))] \approx 0.$$

We would expect that with an external field, R(x, x') is usually positive, so we should not expect the latter to be true.

## **3** Lower bound for $F_N(\beta)$

One easy lower bound is given by conditioning on  $\mathbf{G}_N^{(1)}$  and considering a 1/N-width neighborhood about its orthogonal complement. Letting  $(\mathbf{G}_N^{(1)})^{\perp}$  be the subset of  $S_N$  orthogonal to  $\mathbf{G}_N^{(1)}$ , we then have

$$F_N(\beta) = F_\beta(H_N; S_N) \ge F_\beta(H_N; (\mathbf{G}_N^{(1)})^{\perp}) \ge \beta^2 \xi_0(1)/2 - o(1),$$

where

$$\xi_0(R) = \sum_{p=2}^{P} \gamma_p^2 R^p = \xi(R) - \xi'(0)R$$

and  $\beta \leq \beta_0(\xi)$ . However this bound is suboptimal.

A second approach would be to find a subsphere correlated with  $\mathbf{G}_N^{(1)}$ .

**Definition 3.1.** The band centered at *x* is

Band(
$$x$$
) = { $y \in S_N : \langle x, x \rangle = \langle y, x \rangle$ }.

By fixing  $\mathbf{G}_N^{(1)}$ , we have

$$\operatorname{Band}(\sqrt{q}\mathbf{G}_N^{(1)}) = \left\{\sqrt{q}\tilde{\mathbf{G}}_N^{(1)} + \sqrt{1-q}z : z \in S_N, z \perp \mathbf{G}_N^{(1)}\right\},\$$

where  $\tilde{\mathbf{G}}_N^{(1)} = \frac{\sqrt{N}\mathbf{G}_N^{(1)}}{\|\mathbf{G}_N^{(1)}\|}$  has norm  $\sqrt{N}$ . It follows that

$$F_{\beta}(H_N; S_N) \ge F_{\beta}(H_N; \operatorname{Band}(\sqrt{q}\mathbf{G}_N^{(1)})) = \frac{\beta}{N} H_N(\sqrt{q}\tilde{\mathbf{G}}_N^{(1)}) + F_{\beta}(H_N - H_N(\sqrt{q}\tilde{\mathbf{G}}_N^{(1)}); \operatorname{Band}(\sqrt{q}\tilde{\mathbf{G}}_N^{(1)})),$$

due to the identity

$$F_N(\beta) = \frac{1}{N} \log \int e^{\beta H_N(x)} \, dx = \beta H_N(x_*(q)) / N + \frac{1}{N} \log \int e^{\beta (H_N(x) - H_N(x_*(q)))} \, dx$$

for any  $x_*$ . The first term on the RHS is  $\beta \sqrt{q}N$ , but the second term is intractable, even though  $\tilde{\mathbf{G}}_N^{(1)}$  will cancel out from subtraction. Ultimately, the reason we still cannot solve the second term is that  $\nabla_{\text{sph}}(H_N(\sqrt{q}\tilde{\mathbf{G}}_N^{(1)})) \neq 0$ .

Lemma 3.2. For 
$$y = \sqrt{q} \tilde{\mathbf{G}}_N^{(1)} + \sqrt{1-q}z$$
,  $y' = \sqrt{q} \tilde{\mathbf{G}}_N^{(1)} + \sqrt{1-q}z'$ , where  $z, z' \in S_N$  and  $z, z' \perp \sqrt{q} \tilde{\mathbf{G}}_N^{(1)}$ ,  

$$\mathbb{E}\Big[\Big(H_N(y) - H_N(\sqrt{q} \tilde{\mathbf{G}}_N^{(1)})\Big)\Big(H_N(y') - H_N(\sqrt{q} \tilde{\mathbf{G}}_N^{(1)})\Big] = N\Big(\xi(R(y, y') - \xi(q))\Big) = N\Big(\xi_0(q + (1-q)R(z, z')) - \xi_0(q)\Big)$$

Proof. We have

$$\mathbb{E}H_N(y)H_N(y') = N\xi_0(R(y, y'))$$

and

$$R\left(y,\sqrt{q}\tilde{\mathbf{G}}_{N}^{(1)}\right) = R\left(\sqrt{q}\tilde{\mathbf{G}}_{N}^{(1)}y'\right) = R\left(\sqrt{q}\tilde{\mathbf{G}}_{N}^{(1)},\sqrt{q}\tilde{\mathbf{G}}_{N}^{(1)}\right) = q.$$

Note that we can view  $\tilde{R}(y, y') = R(z, z')$  as the "effective covariance on the band", so that

$$\tilde{\xi}(\tilde{R}) = \xi_0(q + (1-q)\tilde{R}) - \xi_0(q) \Longrightarrow \tilde{\xi}'(0) = \xi_0'(q)(1-q).$$

It seems like we require two conditions to hold in order to find a good band centered at *x* :

• 
$$R(x, \tilde{\mathbf{G}}_N^{(1)}) > 0$$

• 
$$\nabla_{\mathrm{sph}}(H_N(\sqrt{q}\tilde{\mathbf{G}}_N^{(1)})) = 0.$$

Denote  $x_*(q) = \operatorname{argmax}_{x \in \sqrt{q}S_N} H_N(x)$ , so that

$$F_{\beta}(H_N; S_N) \ge F_{\beta}(H_N; \text{Band}(x_*(q)) = \frac{\beta}{N} H_N(x_*(q)) + F_{\beta}(H_N - H_N(x_*(q)); \text{Band}(x_*(q)).$$
(1)

Recall that for  $\beta \leq \beta_0(\xi)$ ,  $q \leq q_0(\xi)$ , if  $\xi'(1) > \xi''(1)$ , then topologically trivially, the ground state energy satisfies GS  $\approx \sqrt{\xi'(1)}$ . We need this to see  $\tilde{\xi}_q(R) = \xi(qR)$ . In addition,

$$\tilde{\xi}'_q(1) = q\xi'(q) \ge q\xi'(0) = q\gamma_1^2 \ge \Omega(q), \qquad \tilde{\xi}''_q(1) = a^2\xi''(q) \le O(q^2)$$

for  $\gamma_1 > 0$ . Equipped with these results, we can observe that the first term in the RHS of equation (1) is

$$\frac{\beta}{N}H_N(x_*(q)) = \beta \sqrt{q\xi'(q)},$$

due to the fact that if q is small and  $\gamma_1 > 0$ , so  $H_N$  is topologically trivial on  $S_N$ . We are now able to modify Lemma 4.1, replacing each  $\sqrt{q}\tilde{\mathbf{G}}_N^{(1)}$  with  $x_*(q)$ :

**Lemma 3.3.** For 
$$y = \sqrt{q}x_*(q) + \sqrt{1-q}z$$
,  $y' = \sqrt{q}x_*(q) + \sqrt{1-q}z'$ , where  $z, z' \in S_N$  and  $z, z' \perp \sqrt{q}x_*(q)$ ,  

$$\mathbb{E}[(H_N(y) - H_N(x_*(q)))(H_N(y') - H_N(x_*(q)))] = \tilde{\xi}(\tilde{R}) - \xi'(0)\tilde{R}.$$

Since  $\xi_q(\tilde{R})$  has no external field, then the second moment method works.

**Theorem 3.4.**  $\liminf_{N\to\infty} F_N(\beta) \ge \max_{q \le q_0(\xi)} \beta \sqrt{q\xi'(q)} + \frac{\beta^2}{2}\xi_q(1) + \frac{1}{2}\log(1-q).$ 

*Proof.* Here is a sketch. We appeal to the idea of the proof of Kac-Rice. Because q is small, then we have topological trivialization on  $\sqrt{q}S_N$ , which implies

$$\frac{1}{N}\log\mathbb{E}|\operatorname{Crt}_{\sqrt{q}S_N}(H_N)| \le o(1).$$

Then,

$$\frac{1}{N}\log\mathbb{E}|\operatorname{Crt}_{\sqrt{q}S_N}(H_N;I_{\epsilon})| \le -\delta \le 0,$$

where

$$I_{\epsilon}(x) = \left\{ \left| F_{\beta}(H_N - H_N(x), \text{Band}(x)) - \beta^2 \xi_q(1)/2 \right| \ge \epsilon \right\}$$

Thus

$$\mathbb{P}[I_{\epsilon}(x)|\nabla_{\mathrm{sph}}(H_N(x)=0] \le e^{-\delta N}.$$

In other words,

 $\mathbb{E}[\text{number of critical points } x \in \sqrt{q} S_N \text{ where } F_{\beta}(H_N; \text{Band}(x)) \text{ is too small}] \leq e^{-\delta N},$ so  $\text{Band}(x_*(q))$  has the correct free energy  $\beta^2 \xi_q(1)/2$ .

The moral of the story is that at high tempeartures with an external field,

- $\mu_{\beta}$  is replica-symmetric on a well-chosen band.
- · Also in certain regimes for tensor PCA, non-spherical models.

## 4 **Upper bound for** $F_N(\beta)$

We now briefly outline an upper bound for the free energy. This is the replica-symmetric case of the Parisi formula.

**Theorem 4.1.**  $F_{\beta}(\xi) \le \min_{q} \left( \frac{\beta^2}{2} (\xi(1) - \xi(q)) + \frac{q}{2(1-q)} + \frac{\log(1-q)}{2} \right)$ 

This matches the lower bound at  $\beta \leq \beta_0(\xi)$ .

*Proof.* We interpolate  $H_N \mapsto \text{linear } L_N \sim \xi_{\text{Lin}}(R) = \xi'(q)R$ . Denoting  $H_{N,\theta}(x) = \cos(\theta)H_N(x) + \sin(\theta)L_N(x)$ , we have

$$\frac{d}{d\theta} \mathbb{E}F_{N,\theta} = \beta \sin\theta \cos\theta \mathbb{E}_{x,x'\sim\mu_{\beta,\theta}} [\xi(R(x,x')) - \xi(1) + \xi'(q)(1 - R(x,x'))].$$

The inner expression on the RHS is minimized at R(x, x') = q, (at least for  $\xi$  which is convex on [-1, 1]). Thus it is lower-bounded by  $\xi(q) - \xi(1) + \xi'(q)(1 - q)$ . Thus

$$F_{\beta}(\xi) \leq \frac{\beta^2}{2}(\xi(1) - \xi(q) - (1 - q)\xi'(q)) + \mathbb{E}F_{\beta}(L_N).$$

By some algebra, one can show that

$$\min_{q} \left( \frac{\beta^2}{2} (\xi(1) - \xi(q) - (1 - q)\xi'(q)) + \mathbb{E}F_{\beta}(L_N) \right) = \min_{q} \left( \frac{\beta^2}{2} (\xi(1) - \xi(q)) + \frac{q}{2(1 - q)} + \frac{\log(1 - q)}{2} \right),$$

which completes the proof.