# Statistics 291: Lecture 13 (March 3rd, 2024)

## Shattering II

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## 1 Non-monotonicity of Franz-Parisi potential

Recall the Franz-Parisi potential

$$\mathscr{F}_\beta(\mathrm{Band}(q\sigma)) = \frac{1}{N}\log\int_{\mathrm{NBHD}_{1/N}(\mathrm{Band}(q\sigma))} e^{\beta H_{N,p}(x)}dx$$

where $\mathrm{NBHD}_\mathrm{r}(S)$ denotes the $r$-neighborhood of a set $S$.

We will prove the following non-monotonicity result from last time.

*Proposition* 1.1. There exists some constant $p_0, \tilde{C}, \bar{C}$ such that for $p \geq p_0$, $\beta \in [\tilde{C}, \sqrt{(1/2 - o(1))\log p})$, $\tilde{C} \gg \bar{C} \gg 1$ such that

$$\frac{d}{dq}\mathbb{E}^{\mathbb{Q}_N}\mathscr{F}_\beta(\mathrm{Band}(q\sigma)) \geq \frac{\beta^2 p}{10} > 0, \quad \forall q \in \left(1-\frac{1}{2p}, 1-\frac{\bar{C}}{\beta p}\right).$$

## 2 Decompose the potential

Recall $\mathbb{Q}$ denotes measure of a planted model with Hamiltonian induced by disorder $G = \frac{\beta}{N^{\frac{p-1}{2}}}\sigma^{\otimes p} + W$ where $\sigma$ is a sample uniformly drawn from sphere $S_N$ and $W$ is an independent Gaussian tensor. In the planted model, an exact formula for $\mathbb{E}^{\mathbb{Q}_N}F_\beta(\mathrm{Band}(q\sigma))$ is available. To describe the this formula, we first define mixture functions

$$\xi(R) = R^p, \qquad \xi_q(R) = \left(q^2 + \left(1-q^2\right)R\right)^p - q^{2p}.$$

and the general centered Gaussian process $H_N$ with covariance

$$\mathbb{E}\left[H_N(x)\,H_N(\tilde{x})\right] = N\xi_q\left(\langle x, \tilde{x}\rangle / N\right)$$

and the associated free energy

$$F_\beta(\xi_q) := \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log\int e^{\beta H_N(\boldsymbol{\sigma})}\mathrm{d}\mu_0(\boldsymbol{\sigma}).$$

Under measure $\mathbb{Q}$, we may decompose Franz-Parisi potential to the following

$$\mathrm{F}_\beta(\mathrm{Band}(q\sigma)) \overset{d}{=} \underbrace{F_\beta\left(\xi_q\right)}_{\text{effective covariance on band}} + \underbrace{\beta^2 q^p}_{\text{spike}} + \underbrace{\frac{1}{2}\log\left(1-q^2\right)}_{\text{volume}} + O(N^{-1/2}). \tag{1}$$

To obtain this relation we write

$$\mathscr{F}_\beta(\text{Band}(q\sigma)) = \frac{\beta}{N} H_{N,p}(q\sigma) + \frac{1}{N}\log\int_{\text{NBHD}_{1/N}(\text{Band}(q\sigma))} e^{\beta(H_{N,p}(x)-H_{N,p}(q\sigma))}dx$$

where the first term satisfies

$$\frac{\beta}{N} H_{N,p}(q\sigma) = \beta^2 q^p + \frac{\beta}{N^{(p+1)/2}}\left\langle W, \sigma^{\otimes p}\right\rangle = \beta^2 q^p + O(N^{-1/2}).$$

and second term give rises to the $F_\beta(\xi_q)$ term via effective covariance arguments covered before (or see Proposition 3.7 in Mark's paper "Shattering in Pure Spherical Spin Glasses").

The idea is that the positivity of the Franz-Parisi potential derivative will come from the spike component and the analysis boils down to controlling derivative of $F_\beta(\xi_q)$, free energy of a mixed-p spin glass with mixture function $\xi_q$. We write Hamiltonian of this mixed-p spin glass as

$$H_N(x) = \sum_{j=1}^{p} \gamma_j H_{N,j}(x)$$

where $H_{N,j}(\sigma) = \frac{1}{N^{(j-1)/2}} \sum_{i_1,\ldots,i_p=1}^{N} g_{i_1\ldots i_p} \sigma_{i_1}\cdots\sigma_{i_p}$.

# 3 Take derivatives

*Lemma* 3.1. For any $N, \beta$, and $H_N(x) = \sum_{j=1}^{p} \gamma_j H_{N,j}(x)$, we have that for $x, \tilde{x} \overset{iid}{\sim} \mu_\beta(H_N)$. Then,

$$\frac{d}{d\gamma_j}\mathbb{E}F_\beta(H_N) = 2\beta^2/j\cdot\left(1 - \mathbb{E}[R(x,\tilde{x})^j]\right).$$

*Proof Sketch.* For fixed disorder,

$$\begin{aligned}\mathbb{E}\frac{d}{d\gamma_j}F_\beta(H_N) &= \mathbb{E}\frac{1}{N}\frac{\frac{d}{d\gamma_j}\int_{S_N} e^{\beta H_N(x)}dx}{Z_\beta(H_N)} = \frac{1}{N}\mathbb{E}\frac{\int_{S_N}\beta H_{N,j}(x)\exp\big(\beta H_N(x)\big)dx}{Z_\beta(H_N)}\\ &= \beta\sum_{i_1,\ldots,i_p=1}^{N}\mathbb{E}g_{i_1,\ldots,i_p}\frac{\int_{S_N} x_{i_1}\ldots x_{i_p} e^{\beta H_N(x)}dx}{N^{(p+1)/2}Z_\beta(H_N)}.\end{aligned}$$

The result then follows from Gaussian integration by parts similarly to previous lectures. □

A simple calculation yields

$$\frac{\mathrm{d}}{\mathrm{d}q}\xi_q(x) = 2pq\left((1-x)\left(q^2+\left(1-q^2\right)x\right)^{p-1} - q^{2p-2}\right).$$

The following Corollary follows from this, the Lemma above and chain rule.

*Corollary* 3.2. For $x, \tilde{x} \overset{iid}{\sim} \mu_\beta(H_N)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}q}F_\beta\left(\xi_q\right) &= \frac{\beta^2}{2}\cdot\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}q}\left(\xi_q(1)-\xi_q(R(x,\tilde{x}))\right)\right]\\ &= -\beta^2 pq\cdot\mathbb{E}\left[(1-R(x,\tilde{x}))\left(q^2+\left(1-q^2\right)R(x,\tilde{x})\right)^{p-1}\right].\\ \frac{\mathrm{d}}{\mathrm{d}\beta}F_\beta\left(\xi_q\right) &= \beta\cdot\mathbb{E}\left[\xi_q(1)-\xi_q(R(x,\tilde{x}))\right] \geq \beta\cdot\mathbb{E}\left[(1-R(x,\tilde{x}))\xi_q'(R(x,\tilde{x}))\right]\\ &= \beta p(1-q^2)\cdot\mathbb{E}\left[(1-R(x,\tilde{x}))\left(q^2+\left(1-q^2\right)R(x,\tilde{x})\right)^{p-1}\right].\end{aligned}$$

## 4 Two tricks

We will employ two tricks

- Bound $\frac{\mathrm{d}}{\mathrm{d}q} F_\beta\left(\xi_q\right)$ by $\frac{\mathrm{d}}{\mathrm{d}\beta} F_\beta\left(\xi_q\right)$
- Bound $\frac{\mathrm{d}}{\mathrm{d}\beta} F_\beta\left(\xi_q\right)$ by ground state energy.

Concretely,

$$\frac{\mathrm{d}}{\mathrm{d}q} F_\beta\left(\xi_q\right) \geq -\frac{\beta q}{1-q^2} \frac{\mathrm{d}}{\mathrm{d}\beta} F_\beta\left(\xi_q\right) \geq -\frac{\beta q}{1-q^2} \mathbb{E} \max_{x \in S_N} \frac{H_N(x)}{N}.$$

Finally, recall that we showed via chaining argument

$$\mathbb{E} \max_{x \in S_N} H_{N,j}(x)/N \leq O\left(\sqrt{j \log j}\right)$$

and by Binomial expansion, using basic inequality $\binom{p}{j} \leq p^j / j!$, $p(1-q^2) \leq 1$ (since $1 \geq 1-q/2p$),

$$\xi_q(R) = \sum_{j=1}^{p} \binom{p}{j} (1-q^2)^j q^{2(N-j)} R^j \leq \sum_{j=1}^{p} \frac{R^j}{j}.$$

Combining these, we conclude

$$\mathbb{E} \max_{x \in S_N} H_N(x)/N \leq C \sum_{j=1}^{p} \frac{\sqrt{j \log j}}{j!} \leq C'$$

where $C'$ is some absolute constant. For $1-(2p)^{-1} < q \leq 1 \leq C\beta$, we conclude from (1) and the above that

$$\frac{\mathrm{d}}{\mathrm{d}q} \mathscr{F}_\beta(\mathrm{Band}(q\sigma)) \geq -\frac{(C\beta+1)q}{1-q^2} + \beta^2 p q^{p-1} > -\frac{C\beta}{1-q} + \frac{\beta^2 p}{2}.$$

By inspection, the right-most expression is positive whenever

$$1 - \frac{1}{2p} \leq q \leq 1 - \frac{2C}{\beta p}, \quad \max\left\{C^{-1}, 4C\right\} \leq \beta.$$

This concludes the proof.