Statistics 291: Lecture 21 (April 9, 2024)

Upper Bound: Guerra's Interpolation Bound

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The goal for today is to prove the interpolation upper bound for the Spherical Parisi Formula. This is work from [Guerra 03, Talagrand 06].

We will assume today that $\xi''(R) \geq 0$ on $|R| \leq 1$. A lot of earlier proofs of the Parisi Formula have this restriction and we assume it to avoid some oddities for the sake of lecture.

1 Review of Ruelle Probability Cascades

Recall the Ruelle Cascades from last time. Given $m_0 < m_1 < \cdots < m_{r-1}$, look at a depth r rooted tree with vertices labelled by $\mathbb{N}^{\varnothing} \cup \mathbb{N}^1 \cup \cdots \cup \mathbb{N}^r = \mathbb{N}^{\leq r}$. There is some room $\mathbb{N}$, with some children including $\mathbb{N}^1$, and then eventually down in the leaves $\mathbb{N}^r$. There are weights on the edges that are i.i.d. from the Poisson point process from last class. For edges on the first layer, we have

$$u_1 \geq u_2 \geq \cdots \sim \mathrm{PPP}(u^{-1-m_0}).$$

In general, if $\gamma = \gamma_1, \ldots, \gamma_d \in \mathbb{N}^d$, then

$$u_{\gamma 1} \geq u_{\gamma 2} \geq \cdots \sim \mathrm{PPP}(u^{-1-m_d}).$$

These are independent for different $\gamma \in \mathbb{N}^{\leq r-1}$. Additionally, let the weights of the leaves be the product of all the edges going down to that leaf. So, we have

$$w_\alpha = \boldsymbol{u}_{\alpha_1} u_{\alpha_1 \alpha_2} \cdots u_{\alpha_1 \cdots \alpha_r} \in \mathbb{N}^r.$$

In general, we have

$$v_\alpha = \frac{w_\alpha}{\sum_{\alpha' \in \mathbb{N}^r} w_{\alpha'}}$$

as our random probability measure on $\mathbb{N}^r$.

Given increasing $\phi : [0,1] \to \mathbb{R}_+$,

$$g_\phi = \{g_\phi(\alpha)\}_{\alpha \in \mathbb{N}^r}$$

is a centered Gaussian process. Then, we have

$$\mathbb{E}[g_\phi(\alpha) g_\phi(\alpha')] = \phi(g_{\alpha \wedge \alpha'}).$$

Explicitly, one can write this Gaussian process as

$$g_\phi(\alpha) = \sum_{i=1}^{r} \sqrt{\phi(g_d) \cdot \phi(g_{d-1})} \cdot \tilde{g}_{\alpha_1 \cdots \alpha_d},$$

where the latter terms $\tilde{g}$ are i.i.d. $\mathcal{N}(0,1)$. We specifically will use

$$\theta(q) = q\xi^{'}(q) - \xi(q),$$

which is increasing since $\theta'(q) = q\xi^{''}(q) \geq 0$ for $q \geq 0$.

2 Interpolation Upper Bound

Define the Hamiltonian

$$H_{N,t}(\sigma, \alpha) = \sin(t)[H_N(\sigma) + t g_\theta(\alpha)] + \cos(t)\langle G_{\xi'}(\alpha), \sigma\rangle.$$

We also define

$$Z_{N,t} = \sum_{\sigma} v_\alpha \int_{S_N} e^{H_{N,t}(\sigma,\alpha)}\, d\sigma = \int e^{H_{N,t}(\sigma,\alpha)}\, d\sigma\, dv(\alpha)$$

and

$$f_N(t) = \frac{1}{N}\mathbb{E}[\log Z_{N,t}].$$

Observe that at $t = \frac{\pi}{2}$, the σ and α parts decouple to give

$$f_N(\pi/2) = \mathbb{E}[F_N(H_N)] + \frac{1}{N}\mathbb{E}\big[\log \sum_{\alpha} v_\alpha e^{g_\theta(\alpha)}\big].$$

In this, we want to find the first term and will be able to compute the second one.

Then, observe that at $t = 0$, we have a mixture of external fields.

Proposition 2.1. *For f as defined above,*

$$f_N'(t) \leq 0 \implies f_N(0) \geq f_N(\frac{\pi}{2}).$$

Proof. Taking the derivative, we have

$$f_N'(t) = -\frac{\sin(t)}{N} \cdot \sum_{j=1}^{N} \mathbb{E} \frac{\sum_d \int \sqrt{\xi'(q_d)\cdot \xi'(q_{d-1})} \cdot \tilde{g}_{\xi',j} \sigma_j e^{H_{N,t}(\sigma,\alpha)}\, d\sigma\, dv(\alpha)}{Z_{N,t}}.$$

Applying Gaussian integration by parts, we can cancel out the $\tilde{g}$ term in the numerator and add an expectation up front in the summation. Differentiating the numerator,

$$-\sin(t)\cos(t) \cdot \frac{\int \big(\xi'(q_d) - \xi'(q_{d-1})\big) \cdot \sigma^2 e^{H_{N,t}(\sigma,\alpha)}\, d\sigma\, dv(\alpha)}{=} \sum_d \xi'(q_d) - \xi'(q_{d-1}) = \xi'(1).$$

Then, differentiating the denominator (a bit more annoying), we have

$$\frac{\int\int \big(\xi'(q_d) - \xi'(q_{d-1})\big) \cdot \sigma_j \sigma_j' 1_{\alpha_1\cdots\alpha_d = \alpha_1'\cdots\alpha_d'} e^{H_{N,t}(\sigma,\alpha) + H_{N,t}(\sigma',\alpha')}\, d\sigma\, d\sigma'\, d\alpha\, d(\alpha')}{(Z_{N,t})^2}.$$

We end up with the overlap function

$$\mathbb{E}_{(\sigma,\alpha),(\sigma',\alpha') \overset{\text{i.i.d.}}{\sim} \mu_{N,t}} [R(\sigma,\sigma')\xi'(q_{\alpha\wedge\alpha'})].$$

Proceeding similarly for the other terms, we end up with

$$f_N'(t) = -\sin(t)\cos(t)\mathbb{E}_{(\sigma,\alpha),(\sigma',\alpha')}\left[\xi'(1) - \xi(1) - \Theta(q_r = 1) - \xi'(q)R + \xi(R) + \Theta(q)\right].$$

The first three terms go to 0 by the definition of Θ. To compute the second line, we fix q. Then, the second line is convex in $R \in [-1,1]$ (since we made the extra assumption $\xi''(R) \geq 0$ for $R \in [-1,1]$). By inspection it thus has a minimum when $q = R$. This minimum value is 0 (by definition of θ again) and so we know that this second line is nonnegative. □

It will be convenient to use the notations

$$\alpha[d] = (\alpha_1, \alpha_1\alpha_2, \ldots, \alpha_1\alpha_2\ldots\alpha_d), \qquad \alpha[1:d] = (\alpha[1], \ldots, \alpha[d]).$$

Next, we have

$$\frac{1}{N}\mathbb{E}\log\sum_\alpha v_\alpha e^{g_\Theta(\alpha)}$$

where

$$e^{g_\Theta(\alpha)} = \prod_d \exp\left(\sqrt{\Theta(q_d) - \Theta(q_{d-1})}\cdot\tilde{g}_{\alpha[d]}\right) = F_r\left(\tilde{g}_{\alpha[1:r]}\right).$$

Using the general property of Ruelle cascades from last time,

$$\frac{1}{N}\mathbb{E}\log\sum_\alpha v_\alpha F_r(\tilde{g}_{\alpha[1:r]}) = \frac{1}{N}\log F_0$$

where by backwards recursion we define:

$$F_d(\tilde{g}[1:d]) = \mathbb{E}\left[F_{d+1}^{m_d}\left(\tilde{g}_{[1:d+1]}\right) \mid \tilde{g}_{[1:d]}\right]^{\frac{1}{m_d}}.$$

Doing out one step of this computation, we have

$$F_{r-1}(\tilde{g}_{\alpha[1:r-1]}) = \prod_{d=1}^{r-1}\exp\left(\sqrt{\theta(q_d) - \theta(q_{d-1})}\tilde{g}_{\alpha[d]}\right)\cdot\mathbb{E}\left[\exp(m_{r-1}\sqrt{\theta(q_r) - \theta(q_{r-1})}\cdot\tilde{g}_\alpha)\right]^{\frac{1}{m_{r-1}}},$$

where the second term becomes

$$e^{m_{r-1}\left(\frac{\theta(q_r)-\theta(q_{r-1})}{2}\right)}.$$

Doing this computation r times, in each step we similarly act on another term of the product defining F_r. Thus we find:

$$\frac{1}{N}\mathbb{E}\log\sum_\alpha v_\alpha e^{g_\Theta(\alpha)} = \sum_{d=1}^r \frac{1}{m_{d-1}}\left(\Theta(q_d) - \Theta(q_{d-1})\right).$$

Finally, we also need $f_N(0)$ which is a bit more complicated. We consider

$$\frac{1}{N}\mathbb{E}\log\sum_\alpha v_\alpha \int_{\mathbb{R}^N} e^{\langle G_\xi(\alpha),\sigma\rangle}\, d\lambda_b(\sigma)$$

where one replaces S_N by $d\lambda_b(\sigma) = \mathcal{N}(0, I_N/b)$ for $b > 1$. The point is that to leading exponential order,

$$\mathbb{P}_{x\sim\lambda_b}[\|x\| \in [\sqrt{N}, \sqrt{N} + \frac{1}{N}]] \approx \sqrt{\frac{b}{2\pi}}^N e^{-Nb/2}\cdot\mathrm{Vol}(S_N) = \exp\left(N\left(\frac{1 + \log b - b}{2}\right)\right).$$

Therefore modulo some slight technicalities, we can prove an upper bound for any desired λ_b and transfer to the sphere up to this additional term. λ_b reduces this to a scalar problem. The advantage of working with λ_b is that the N coordinate directions behave independently in the backward recursion, so we will just focus on one of them. Fix 1 coordinate $j \in [N]$. Then, we have

$$F_r(\tilde{g}_{[1:r]}) = \sqrt{\frac{b}{2\pi}} \int_{\mathbb{R}} \exp\left(\frac{-bz^2}{2} + g_{\xi'}(\alpha)z\right) dz = \exp\left(\frac{g_{\xi'}(\alpha)^2}{2b}\right),$$

which (doing one step) implies

$$F_{r-1}(\tilde{g}_{\xi}(\alpha_{[1:r]})) = \mathbb{E}\left[\exp\left(\frac{m_{r-1}[g_{\xi'}(\alpha_{[1:r-1]}) + \sqrt{\xi'(q_r) - \xi'(q_{r-1})} \cdot z]^2}{2b}\right)\right]^{\frac{1}{m_{r-1}}}.$$

In general, for z a standard Gaussian, one easily computes

$$\mathbb{E}^z[e^{(a_1+a_2z)^2}] = e^{(1-2a_2^2)^{-1}}\sqrt{1-2a_2^2}.$$

Plugging in the corresponding values of a_1 and a_2, we obtain

$$F_{r-1}(\tilde{g}_{\xi}(\alpha_{[1:r]})) = \exp\left(\frac{g_{\xi'}(\alpha_{[1:r-1]})^2}{2(b - m_{r-1}(\xi'(q_r) - \xi'(q_{r-1})))}\right) \cdot \left(\frac{b - m_{r-1}(\xi'(q_r) - \xi'(q_{r-1}))}{b}\right)^{\frac{1}{2m_{r-1}}}.$$

The first term is of a similar form as F_r, while the second is a constant. Recursing, we will continue to pick up more constant factors of the latter type. The end result is:

$$\limsup_{N\to\infty} \mathbb{E}F_N \leq \frac{1}{2}\left(\frac{\xi'(q_0)}{D_r} + \sum_d \frac{\log(D_d/D_{d-1})}{m_{d-1}} + b - 1 - \log b\right),$$

where

$$D_k = b - \sum_{d=1}^{k} m_{d-1}(\xi'(q_d) - \xi'(q_{d-1}))$$

So, we get the Spherical Parisi Upper Bound

$$\limsup_{N\to\infty} \mathbb{E}F_N(H_N) \leq \inf_{b>1;\vec{m},\vec{q}} \frac{1}{2}\left(\frac{\xi'(q_0)}{D_r} + \sum_d \frac{\log(D_d/D_{d-1})}{m_{d-1} - \sum_d m_d(\theta(q_d) - \theta(q_{d-1}))} + b - 1 - \log(b)\right).$$

Next time, we'll make this formula slightly nicer by thinking of things in a less discrete way and also present some ideas about the lower bound.